A BRIEF EXCURSION INTO THE ROLE OF FUNDAMENTAL RELATIONS IN ALGEBRAIC HYPERSTRUCTURES

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Abstract. The overall aim of this paper is to present an introduction to some of the results, methods and ideas about fundamental relations on algebraic hyperstructures . Fundamental relations are special kind of strongly regular relations and they are important in the theory of algebraic hyperstructures . Indeed, the main tools connecting the class of hyperstructures with the classical algebraic structures are the fundamental relations . In this paper, we study :
(1) Congruence relation on semigroups;
(2) Congruences of groups and normal subgroups;
(3) Regular and strongly regular relations in (semi)hypergroups;
(4) The fundamental relation β^*;
(5) The relation Γ^*;
(6) Solvable and nilpotent hypergroups (polygroups);
(7) Strongly regular relations in ordered semihypergroups;
(8) Fundamental relations in hyperrings and H_v-rings;
(9) α^*-relations and fundamental commutative ring .

1. Congruence relation on semigroups

Let us firstly recall some definitions on semigroups . Suppose that S is a semigroup . Then, an equivalence relation ρ on S is said to be a *left* (*right*) *compatible* if $a\rho b$ and $x \in S$ imply $xa\rho xb$ ($ax\rho bx$), and is said to be *compatible* if it is both left and right compatible . A compatible relation on a semigroup S is called a *congruence relation* . It is easy to see that an equivalence relation ρ on a semigroup S is a congruence if and only if $a\rho b$ and $c\rho d$ imply $ac\rho bd$. Let S/ρ be the quotient set and $\pi : S \to S/\rho$ the canonical mapping which maps a to $\rho(a)$, where $\rho(a)$ is the equivalence class of S containing a . The following results are easy to prove (see also [3]) . Let S be a semigroup and ρ an equivalence relation on S . Then, the following statements are equivalent :

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(1) The relation ρ is a congruence relation;
(2) The set S/ρ becomes a semigroup, which is called *quotient semigroup*, equipped with the multiplication defined by $\rho(x)\rho(y) = \rho(xy)$.

If the quotient semigroup S/ρ admits a unit element $1_{S/\rho}$, then $Ker\pi = 1_{S/\rho}$ and $Ker\pi$ is a sub-semigroup, where $\pi : S \to S/\rho$ is the canonical homomorphism . Let S and T be two semigroups and $f : S \to T$ be a semigroup homomorphism . We define $Kerf = \{(a,b) \in S \times S \mid f(x) = f(y)\}$, and we call this the *kernel* of the map f . The first isomorphism theorem for semigroups tells us that with every semigroup homomorphism $f : S \to T$, the $Kerf$ is a congruence relation on S and $S/Kerf \cong Imf$. Let I be a non-empty subset of S . Then, I is a *right ideal* if $IS \subseteq I$; and I is a *left ideal* if $SI \subseteq I$. Finally, I is a (*two sided*) *ideal* if $IS \cup SI \subseteq I$. If I is a proper ideal of a semigroup S, then $\rho_I = \{(a,a) \mid a \in S\} \cup (I \times I)$ is a congruence on S . A congruence of this type is called a *Rees congruence* .

2. Congruences of groups and normal subgroups

The definition of a congruence depends on the type of algebraic structure under consideration . In particular, the definition of congruence can be made for groups . The common theme is that a congruence is an equivalence relation on group that is compatible with respect to binary operation of group . Let G be a group with the identity e and ρ be a congruence relation on G . If $N = \rho(e)$, then N is a normal subgroup of G . Indeed, since $e \in N$, N is non-empty . Suppose that $a, b \in N$ are two arbitrary elements . Then, $e\rho a$ and $e\rho b$, and so $ee\rho ab$. Thus, $ab \in N$. Similarly, $a^{-1}e\rho a^{-1}a$ which implies that $a^{-1}\rho e$. Hence, $a^{-1} \in N$. Now, we show that N is a normal subgroup in G . Suppose that $a \in G$ and $n \in N$ are arbitrary elements . We have $e\rho n$. Hence, $aea^{-1}\rho ana^{-1}$. This implies that $e\rho ana^{-1}$. Therefore, $ana^{-1} \in N$.

Let G be a group, ρ be a congruence relation on G and $N = \rho(e)$. Using the congruence property, we see that $a\rho b \Leftrightarrow e = a^{-1}a\rho a^{-1}b \Leftrightarrow a^{-1}b \in N$. This seems to suggest that we can use any normal subgroup to get a congruence . In fact it works as next proposition shows . Let G be a group with a normal subgroup N and define a relation ρ on G by $a\rho b$ if and only if $a^{-1}b \in N$. Then, ρ is a congruence on G and $\rho(a) = aN$. In particular $\rho(e) = N$. Let G be a group, ρ be a congruence relation on G and N be a corresponding normal subgroup . Let $G/N = \{\rho(a) = aN \mid a \in G\}$ with a binary operation $\rho(a) \cdot \rho(b) = \rho(ab)$ (that is $aN \cdot bN = abN$) . Notice that this is well defined as ρ is a congruence relation . Then, G/N is a group with respect to this binary operation .

3. Regular and strongly regular relations in (semi)hypergroups

Hyperstructure theory both extends some well-known group results and introduce new topics leading us to a wide variety of applications, as well as to a broadening of the investigation fields . A comprehensive review of the theory of hyperstructures appears in [5, 7, 8, 25] . Let H be a non-empty set and $\circ : H \times H \to \mathcal{P}^*(H)$ be a hyperoperation . The couple $(H, \circ)$ is called a *hypergroupoid* . For any two nonempty subsets A and B of H and $x \in H$, we define $A \circ B = \bigcup_{a \in A, b \in B} a \circ b$,

$A \circ x = A \circ \{x\}$ and $x \circ B = \{x\} \circ B$. A hypergroupoid $(H, \circ)$ is called a *semihypergroup* if for all a, b, c of H we have $(a \circ b) \circ c = a \circ (b \circ c)$, which means that $\bigcup_{u \in a \circ b} u \circ c = \bigcup_{v \in b \circ c} a \circ v$. A hypergroupoid $(H, \circ)$ is called a *quasihypergroup* if for all a of H we have $a \circ H = H \circ a = H$. This condition is also called the *reproduction axiom* . A hypergroupoid $(H, \circ)$ which is both a semihypergroup and a quasihypergroup is called a *hypergroup* . By using a certain type of equivalence relations, we can connect semihypergroups to semigroups and hypergroups to groups . These equivalence relations are called strong regular relations . More exactly, starting with a (semi)hypergroup and using a strong regular relation, we can construct a (semi)group structure on the quotient set . A natural question arises : Do they also exist regular relations The answer is positive, regular relations provide us new (semi)hypergroup structures on the quotient sets . Let us define these notions . First, we do some notations . Let $(H, \circ)$ be a semihypergroup and R be an equivalence relation on H . If A and B are nonempty subsets of H, then $A\overline{R}B$ means that for all $a \in A$ there exists $b \in B$ such that aRb and for all $b' \in B$ there exists $a' \in A$ such that $a'Rb'$; also $A\overline{\overline{R}}B$ means that for all $a \in A$ and for all $b \in B$, we have aRb . The equivalence relation R is called (1) *regular on the right* (*on the left*) if for all x of H, from aRb, it follows that $(a \circ x)\overline{R}(b \circ x)$ $((x \circ a)\overline{R}(x \circ b)$ respectively); (2) *strongly regular on the right* (*on the left*) if for all x of H, from aRb, it follows that $(a \circ x)\overline{\overline{R}}(b \circ x)$ $((x \circ a)\overline{\overline{R}}(x \circ b)$ respectively); (3) R is called *regular* (*strongly regular*) if it is regular (strongly regular) on the right and on the left .

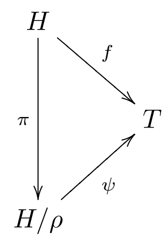
Theorem 3.1. [5, 8] *Let $(H, \circ)$ be a (semi)hypergroup and ρ be an equivalence relation on H .*

(1) *If ρ is regular, then H/ρ is a (semi)hypergroup with respect to the following hyperoperation : $\rho(x) \odot \rho(y) = \{\rho(z) \mid z \in x \circ y\}$.*
(2) *If the above hyperoperation is well defined on H/ρ, then ρ is regular .*

Theorem 3.2. [5, 8] *Let $(H, \circ)$ be a (semi)hypergroup and ρ be an equivalence relation on H .*

(1) *If ρ is strongly regular, then H/ρ is a (semi)group with respect to the following operation : $\rho(x) \odot \rho(y) = \rho(z)$, for all $z \in x \circ y$.*
(2) *If the above operation is well defined on H/ρ, then ρ is strongly regular .*

Let $(H, \circ)$ and $(T, \diamond)$ be two semihypergroups . A function $f : H \to T$ is called an *inclusion homomorphism* if it satisfies $f(x \circ y) \subseteq f(x) \diamond f(y)$, for all $x, y \in H$; and f is called a *strong homomorphism* (for short, *homomorphism*) if $f(x \circ y) = f(x) \diamond f(y)$, for all $x, y \in H$. Let f be a homomorphism from a semihypergroup $(H, \circ)$ into $(T, \diamond)$. The relation fof^{-1} is an equivalence relation ρ on H $(a\rho b \Leftrightarrow f(a) = f(b))$ is known as the *kernel* of f . One can see that $ker f$ is a regular relation (Theorem 3.3 in [10]) . The canonical mapping associated with ρ is $\pi : H \to H/ker f$, where $\pi(a) = \rho(a)$. The mapping $\psi : H/\rho \to T$, where $\psi(\rho(a)) = f(a)$ is then the unique bijection that makes the following diagram commutative [10] .



4. The fundamental relation β^*

The main tools connecting the class of hyperstructures with the classical algebraic structures are the fundamental relations . The fundamental relation has an important role in the study of semihypergroups and especially of hypergroups . For all $n > 1$, we define the relation β_n on a semihypergroup $(H, \circ)$ as follows :

$$x\beta_n y \text{ if there exists } a_1, \ldots, a_n \text{ in } H \text{ such that} \{x, y\} \subseteq \prod_{i=1}^{n} a_i$$

and we set $\beta = \bigcup_{n\geq 1} \beta_n$, where $\beta_1 = \{(x, x) \mid x \in H\}$ is the diagonal relation on H . This relation was introduced by Koskas [19] and studied mainly by Corsini [5], Davvaz [8], Davvaz and Leoreanu-Fotea [12], Freni [15], Vougiouklis [25], and many others . Clearly, the relation β is reflexive and symmetric . Denote by β^* the transitive closure of β . A relation ρ^* is the *transitive closure* of a relation ρ if and only if

(1) ρ^* is transitive,
(2) $\rho \subseteq \rho^*$,
(3) for any relation θ, if $\theta \subseteq \rho$ and θ is transitive, then $\rho^* \subseteq \theta$, i.e., ρ^* is the smallest relation that satisfies (1) and (2) .

For any relation ρ, the transitive closure of ρ always exists .

Theorem 4.1. [5, 8] *Let $(H, \circ)$ is a semihypergroup . Then, β^* is the smallest strongly regular relation on H .*

By using Theorems 3.2 and 4.1, we obtain the following result . Let $(H, \circ)$ be a (semi)hypergroup . Then, the relation β^* is the smallest equivalence relation on H such that the quotient H/β^* is a (semi)group . β^* is called the *fundamental relation* on H and H/β^* is called the *fundamental group* . Let $(H, \circ)$ is a hypergroup and consider the canonical projection $\varphi_H : H \to H/\beta^*$. The *heart* of H is the set $\omega_H = \{x \in H \mid \varphi_H(x) = 1\}$, where 1 is the identity of the group $(H/\beta^*, \otimes)$.

Let $(H, \circ)$ is a semihypergroup and A be a non-empty subset of H. We say that A is a *complete part* of H if for any positive integer n and for all $a_1, \ldots, a_n$ of H, the following implication holds :

$$A \cap \prod_{i=1}^{n} a_i \neq \emptyset \;\Rightarrow\; \prod_{i=1}^{n} a_i \subseteq A.$$

Complete parts were introduced and studied for the first time by Koskas [19]. Later, this topic was analyzed by Corsini [6] and Sureau [23] mostly in the general theory of hypergroups.

Theorem 4.2. [5, 12] *If $(H, \circ)$ is a semihypergroup and ρ is a strongly regular relation on H, then for all $a \in H$, $\rho(a)$ is a complete part of H.*

If A is a subset of H, denote by $C(A)$ the complete closure of A, which is the smallest complete part of H, that contains A. Denote $K_1(A) = A$ and for all $n \geq 1$,

$$K_{n+1}(A)=\Big\{x\in H \mid \exists p \in \mathbb{N}^*, \exists(h_1, \dots, h_p) \in H^p : x \in \prod_{i=1}^{p} h_i, K_n(A) \cap \prod_{i=1}^{p} h_i \neq \emptyset\Big\},$$

and let $K(A) = \bigcup_{n\geq 1} K_n(A)$. We have $C(A) = K(A)$.

Theorem 4.3. [5, 12] *If a is an arbitrary element of a hypergroup $(H, \circ)$, then*

(1) *For all $n \geq 2$ we have $K_n(K_2(a)) = K_{n+1}(a)$.*
(2) *$a \in K_n(b)$ implies $b \in K_n(a)$.*

It is easy to see that the binary relation defined by $xKy \Leftrightarrow \exists n \geq 1,\ x \in K_n(y)$ is an equivalence relation.

Theorem 4.4. [5, 12]

(1) *The equivalence relations K and β^* coincide.*
(2) *If B is a nonempty subset of H, then we have $C(B) = \bigcup_{b\in B} C(b)$.*
(3) *ω_H is a complete part of H.*
4) *Denote the class of all complete parts subhypergroups of H by $CPS(H)$. Then, $\omega_H = \bigcap_{K\in CPS(H)} K$.*
(5) *The relation β is transitive if and only if for all x of H, we have $C(x) = K_2(x)$.*
(6) *For all x of ω_H, we have $K_2(x) = \omega_H$.*

Freni in [15] that for hypergroups, $\beta = \beta^*$. Indeed, we have the following result. If $(H, \circ)$ is a hypergroup, then the relation β is an equivalence relation on H. If H is a semihypergroup, β^* is not equal to β in general, see Remark 82 in [5].

The theory of H_v-structures (or weak hyperstructures) has been introduced by Vougiouklis in 1990 during the fourth AHA congress [24], also see [25]. Vougiouklis defined the notion of an H_v-group and since then many researchers have worked on this new topic of algebra and developed it,for example see [9, 21, 25, 26]. Let $(H, \circ)$ be a hypergroupoid. Then, $(H, \circ)$ is called an H_v-group if (1) $(a\circ b)\circ c \cap a\circ(b\circ c) \neq \emptyset$ for all $a, b, c \in H$; (2) $a \circ H = H \circ a = H$ for all $a \in H$. Let $(H, \circ)$ be an H_v-group. We define the relation β^* on H as the smallest equivalence relation on H such that the quotient H/β^* is a group. The β^* is called the *fundamental relation*. Now, we denote by $\mathcal{U}$ the set of all finite product of elements of H and define the relation β in H as follows :

$$a\beta b \ \Leftrightarrow\ \{a, b\} \subseteq \mathcal{U} \text{ for some } u \in \mathcal{U}.$$

Let $(H, \circ)$ be an H_v-group. The fundamental relation β^* is the transitive closure of the relation β [25].

Open problem 1. Is it true $\beta = \beta^*$ for H_v-groups

5. The relation Γ^*

Freni in [14] introduced the relation Γ as a generalization of the relation β. Let H be a semihypergroup. Then, we set $\Gamma_1 = \{(x, x) \mid x \in H\}$ and for every integer $n > 1$, Γ_n is the relation defined as follows :

$$x\Gamma_n y \ \Leftrightarrow\ \exists(z_1, \ldots, z_n) \in H^n,\ \exists\sigma \in \mathbb{S}_n\ :\ x \in \prod_{i=1}^{n} z_i,\ y \in \prod_{i=1}^{n} z_{\sigma(i)},$$

where $\mathbb{S}_n$ is the symmetric group on n letters. The letter Γ already has been used for the corresponding fundamental relation on hyperrings by Vougiouklis [24, 25]. Thus, there is a confusion on the symbolism. Therefore in this section we use the symbol Γ_H instead of Γ. If $(H, \circ)$ is a commutative hypergroup, then the relation Γ_H is equal to β. Obviously, for $n \geq 1$, the relations Γ_{Hn} are symmetric, and the relation $\Gamma_H = \bigcup_{n\geq 1}(\Gamma_H)_n$ is reflexive and symmetric. Let $(\Gamma_H)^*$ be the transitive closure of Γ_H.

Theorem 5.1. [8, 14] *The relation $(\Gamma_H)^*$ is a strongly regular relation.*

The quotient $H/(\Gamma_H)^*$ is a commutative semigroup. Furthermore, if H is a hypergroup, then $H/(\Gamma_H)^*$ is a commutative group. The relation $(\Gamma_H)^*$ is the smallest strongly regular relation on a semihypergroup H such that the quotient $H/(\Gamma_H)^*$ is commutative semigroup. Let M be a non-empty subsets of H. We say that M is a Γ_H*-part* of H if for all $n \in \mathbb{N}$, for all $(z_1, \ldots, z_n) \in H^n$ and for all $\sigma \in \mathbb{S}_n$, we have

$$\prod_{i=1}^{n} z_i \cap M \neq \emptyset \Rightarrow \prod_{i=1}^{n} z_{\sigma(i)} \subseteq M.$$

Lemma 5.2. [8, 14] *Let M be a non-empty subsets of H. Then, the following conditions are equivalent :*

(1) *M is a Γ_H-part of H;*
(2) *$x \in M,\ x\Gamma_H y \Rightarrow y \in M$;*
(3) *$x \in M,\ x\Gamma_H^* y \Rightarrow y \in M$.*

Let $(H, \circ)$ be a semihypergroup. For all $x \in H$, we set

- $T_n(x) = \Big\{(x_1, \ldots, x_n) \in H^n \mid x \in \prod_{i=1}^{n} x_i\Big\}$;
- $P_n = \bigcup\Big\{\prod_{i=1}^{n} x_{\sigma(i)} \mid \sigma \in \mathbb{S}_n,\ (x_1, \ldots, x_n) \in T_n(x)\Big\}$;
- $P_\sigma(x) = \bigcup_{n\geq 1} P_n(x)$.

For every $x \in H$, $P_\sigma(x) = \{y \in H \mid x\Gamma_H y\}$.

Theorem 5.3. [8, 14] *Let $(H, \circ)$ be a semihypergroup. Then, the following conditions are equivalent :*

(1) *Γ_H is transitive;*
(2) *$(\Gamma_H)^*(x) = P_\sigma(x)$, for all $x \in H$;*
(3) *$P_\sigma(x)$ is a Γ_H-part of H, for all $x \in H$.*

Similar to the fundamental relation β, we have the following result. If $(H, \circ)$ is a hypergroup, then the relation Γ_H is an equivalence relation on H [14].

6. Solvable and nilpotent hypergroups (polygroups)

A special subclass of hypergroups is the class of polygroups. We recall the following definition from [4, 8]. A *polygroup* is a system $P = \langle P, \circ, e, ^{-1} \rangle$, where $\circ : P \times P \to \mathcal{P}^*(P)$, $e \in P$, $^{-1}$ is a unitary operation on P and the following axioms hold for all $x, y, z \in P$: (1) $(x \circ y) \circ z = x \circ (y \circ z)$; (2)] $e \circ x = x \circ e = x$; (3) $x \in y \circ z$ implies $y \in x \circ z^{-1}$ and $z \in y^{-1} \circ x$. The concept of solvable and nilpotent polygroups are studied in [1] and [17], respectively. Also, see [2, 8]. Let H be a hypergroup. We define

(1) $[x, y]_r = \{h \in H \mid x \cdot y \cap y \cdot x \cdot h \neq \emptyset\}$;
(2) $[x, y]_l = \{h \in H \mid x \cdot y \cap h \cdot y \cdot x \neq \emptyset\}$;
(3) $[x, y] = [x, y]_r \cup [x, y]_l$.

From now on we call $[x, y]_r$, $[x, y]_l$ and $[x, y]$ *right commutator* x *and* y*, left commutator* x *and* y and *commutator* x *and* y, respectively. Also, we will denote $[H, H]_r$, $[H, H]_l$ and $[H, H]$ the set of all right commutators, left commutators and commutators, respectively Let X be a non-empty subset of a polygroup $\langle P, \cdot, e, ^{-1} \rangle$. Let $\{A_i \mid \ i \in J\}$ be the family of all subpolygroups of P containing X. Then, $\bigcap_{i \in J} A_i$ is called the *subpolygroup generated* by X. This subpolygroup is denoted by $< X >$ and we have $< X >= \cup\{x_1^{\varepsilon_1} \cdot \ldots \cdot x_k^{\varepsilon_k} \mid x_i \in X, k \in \mathbb{N}, \varepsilon_i \in \{-1, 1\}\}$. If $X = \{x_1, x_2, \ldots, x_n\}$, then the subpolygroup $< X >$ is denoted $< x_1, x_2, \ldots, x_n >$. In a special case $< [P, P]_r >$, $< [P, P]_l >$ and $< [P, P] >$ are shown by P'_r, P'_l and P', respectively.

Proposition 6.1. [1, 8] *Let* $\langle P, \cdot, e, ^{-1} \rangle$ *be a polygroup and* $(x, y) \in P^2$*. Then,*

(1) $[x, y]_r = [x^{-1}, y^{-1}]_l$;
(2) $P' = P'_r = P'_l$;
(3) $x \in P' \Rightarrow x^{-1} \in P'$.

If $\langle P, \cdot, e, ^{-1} \rangle$ is a polygroup, then P' is a subpolygroup of P. From now on we call P' the *derived subpolygroup* of P.

Proposition 6.2. [1, 8] *Let* $\langle P, \cdot, e, ^{-1} \rangle$ *be a polygroup. Then,* $P' = \{e\}$ *if and only if* P *be an abelian group.*

Let $\langle P, \cdot, e, ^{-1} \rangle$ be a polygroup. Then, we define $N(P') = \{u \in P' \mid u \cdot P' = P' \cdot u\}$ and is called the *normalizer* P' in P. Recall that a subhypergroup K of a hypergroup $(H, \cdot)$ is invertible on the right if and only if $K \setminus H = \{K \cdot x \mid x \in H\}$ is a partition of H. If K is an invertible to the left subhypergroup of a hypergroup H, then the quotient $K \setminus H$ by the hyperoperation $K \cdot x \otimes K \cdot y = \{K \cdot z \mid z \in x \cdot K \cdot y\}$, is a hypergroup.

Theorem 6.3. [1, 8] *If* $\langle P, \cdot, e, ^{-1} \rangle$ *is a polygroup, then*

(1) P' *is invertible;*
(2) $N = N(P')$ *is subpolygroup of* $\langle P, \cdot, e, ^{-1} \rangle$*;*
(3) $P' \trianglelefteq N(P)$ *and* $(P' \setminus N, \otimes)$ *is commutative polygroup;*
(4) *if* K *is a complete subpolygroup of* P *such that* $K \setminus P$ *is commutative, then* $P' \subseteq K$.

A polygroup P is called *perfect* if $P' = P$. A polygroup P is called *solvable* if $P^{(n)} = \omega_{_P}$, for some $n \in \mathbb{N}$, where $P^{(1)} = P'$ and $P^{(n+1)} = (P^{(n)})'$. Every non-trivial perfect group is not solvable. One can see the above statement is not true for the class of polygroups [1, 8]. In the following theorem, consider $G//H$ as a double coset algebra.

Theorem 6.4. [1, 8] *Let $(G, \cdot)$ be a group and H be a subgroup of G. We set $HG'H = \{HgH \mid g \in G'\}$. Then,*

(1) $HG'H \subseteq (G//H)'$;
(2) *If $G' \cdot H = G$, then $(G//H)$ is a perfect polygroup;*
(3) *If $HG'H = (G//H)$, then $G' \cdot H = G$.*

Let P be a hypergroup. Suppose that $\tau = \bigcup_{m \geq 1} \tau_m$, where τ_1 is the diagonal relation and for every integer $m > 1$, τ_m is the relation defined as follows :

$$x \tau_m y \;\Leftrightarrow\; \exists (z_1, \ldots, z_m) \in P^m, \exists \sigma \in \mathbb{S}_m : \sigma(i) = i \;\text{ if } z_i \notin P' \text{ such that}$$
$$x \in \prod_{i=1}^{m} z_i \quad \text{and} \quad y \in \prod_{i=1}^{m} z_{\sigma(i)}.$$

Obviously, the relation τ is reflexive and symmetric. Now, let τ^* be the transitive closure of τ. The relation τ^* is a strongly regular relation [1, 8]. Let $(H, \cdot)$ and $(K, *)$ be hypergroups. A map $f : H \to P^*(K)$ is called a *good multihomomorphism* if $f(x \cdot y) = f(x) * f(y)$, for all $x, y \in H$. If $(H, \cdot) = (K, *)$ and $\bigcup_{h \in H} f(h) = H$, then f is called *a generalized automorphism.* Moreover, we will denote by $GAut(H)$ the set of all generalized automorphisms of $(H, \cdot)$.

Proposition 6.5. [1, 8] *$(GAut(H), \circ)$ is a monoid, where $\circ$ is defined as follows : $(f \circ g)(h) = \bigcup_{a \in g(h)} f(a)$, for all $f, g \in GAut(H)$ and $h \in H$. Moreover, $Aut(H)$ (i.e., the group of automorphism of H) is a subgroup of $GAut(H)$.*

Let $(H, \cdot)$ and $(K, *)$ be two hypergroups. We consider the monoid $GAut(H)$ and the group K/τ^*. Let : $\varphi : K/\tau^* \to GAut(H)$ such that $\tau^*(x) \mapsto \varphi_{_{\tau^*(x)}}$ be a homomorphism. Then, we define a hyperoperation in $H \times K$ as follows:$(x_1, y_1) \circ (x_2, y_2) = \{(x, y) \mid x \in x_1 \cdot \varphi_{_{\tau^*(y_1)}}(x_2), y \in y_1 * y_2\}$. We call it a *$\tau$-multisemi-direct hyperproduct* of hypergroups H and K through φ and we denote it by $H \times_\varphi K$ [1, 8]. Moreover, we call a τ-multisemi-direct hyperproduct is special if $Im(\varphi) \subseteq Aut(H)$. Let H and K be two hypergroups. Then, $H \times K$ equipped with the τ-multisemi-hyperproduct is a hypergroup. Let P_1 and P_2 be two polygroups. Then, $P_1 \times P_2$ equipped with a special τ-multisemi-hyperproduct is a ploygroup [1, 8]. The τ-multisemi-direct hyperproduct of polygroups P_1 and P_2 through zero homomorphism $\varphi_{_0}$, i.e., $\varphi_{_0} : P_2/\tau^* \to GAut(P_1)$ with $\varphi_{_0}(\tau^*(x)) = i_{_{Aut(P_1)}}$ which we denote it by $P_1 \times P_2$ and is called *τ-direct hyperproduct* of P_1 and P_2. Let P_1, P_2 be two polygroups. Then, $(P_1 \times P_2)' = P_1' \times P_2'$ [1, 8] Let P_1, P_2 be two polygroups. If $P_1 \times P_2$ is solvable, then P_1 and P_2 are solvable. τ-direct hyperproduct of two polygroups P_1 and P_2 is perfect if and only if P_1 and P_2 are perfect. Let H be a regular hypergroup. For $n \in \mathbb{N}$, let $a_1, \ldots, a_n$ be elements in H, and $a_1', \ldots, a_n'$ are their inverses in H, respectively. The set

$$a_1 \cdot a_2 \cdot \ldots \cdot a_n \cdot a_n' \cdot a_{n-1}' \cdot \ldots \cdot a_1'$$

is called a *product of type zero* and denote with $N(0)$ the union of all products of type 0.

Theorem 6.6. [1, 8] *Let H be a regular and reversible hypergroup and e be a bilateral identity. If $e \in \prod_{i=1}^{n} z_i$, then there exist inverses of $z_1, \ldots, z_n$ respectively $z'_1, \ldots, z'_n$ such that $z_n \in z'_{n-1} \cdot z'_{n-2} \cdot \ldots \cdot z'_1 \cdot e$.*

Theorem 6.7. [1, 8] *If H is a regular and reversible hypergroup, then the heart of H is the union of the products of type zero (i.e., $\omega_H = N(0)$).*

Let $(G, \cdot)$ be a group and $P_G = G \cup \{a\}$, where $a \notin G$. We define on P_G the hyperoperations $\circ$ as follows :

(1) $a \circ a = e$;
(2) $e \circ x = x \circ e = x$, for every $x \in P_G$;
(3) $a \circ x = x \circ a = x$, for every $x \in P_G \setminus \{e, a\}$;
(4) $x \circ y = x \cdot y$, for every $(x, y) \in G^2$ such that $y \neq x^{-1}$;
(5) $x \circ x^{-1} = \{e, a\}$, for every $x \in P_G \setminus \{e, a\}$.

Proposition 6.8. [8, 17] *If G is a group, then $\langle P_G, \circ, e, ^{-1} \rangle$ is a polygroup. Moreover, $P_G/\beta^* \cong G$.*

A polygroup $\langle P, \cdot, e, ^{-1} \rangle$ is said to be *nilpotent* if $\ell_n(P) \subseteq \omega_P$ or equivalently $\ell_n(P) \cdot \omega_P = \omega_P$, for some integer n, where $\ell_0(P) = P$ and

$$\ell_{k+1}(P) = \langle \{h \in P \mid x \cdot y \cap h \cdot y \cdot x \neq \emptyset, \text{ such that } x \in \ell_k(P) \text{ and } y \in P\} \rangle.$$

The smallest integer c such that $\ell_c(P) \cdot \omega_P = \omega_P$ is called the *nilpotency class* or for simplicity the *class* of P. Notice that $P = \ell_0(P) \supseteq \ell_1(P) \supseteq \ell_2(P) \supseteq \ldots$ that is $\{\ell_k(P)\}_{k \geq o}$ is a decreasing sequence which we call it *generalized descending central series*.

Theorem 6.9. [8, 17] *Let $\langle P, \cdot, e, ^{-1} \rangle$ be a polygroup. Then, P is nilpotent if and only if $G = P/\beta^*$ is nilpotent.*

Let G be a group. Then, P_G is nilpotent if and only if G is nilpotent.

Theorem 6.10. [8, 17] *Let $\langle P, \cdot, e, ^{-1} \rangle$ be a polygroup and N be a normal subpolygroup of P. Then, $\ell_n(P/N) = \ell_n(P) \cdot N/N$, for all $n \geq 0$.*

If $\langle P, \cdot, e, ^{-1} \rangle$ is a nilpotent polygroup, then

(1) every subpolygroup of P is nilpotent;
(2) if N is a normal subpolygroup of P, then P/N is nilpotent.

Let $\langle P, \cdot, e, ^{-1} \rangle$ be a polygroup. We define $Z_0(P) = \omega_P$ and $Z_n(P) = \langle \{x | x \cdot y \cdot Z_{n-1}(P) = y \cdot x \cdot Z_{n-1}(P), \forall y \in P\} \rangle$, for all $n \in \mathbb{N}$. Notice that $\omega_P = Z_0(P) \subseteq Z_1(P) \subseteq Z_2(P) \subseteq \ldots$ that is $\{Z_m(P)\}_{m \geq 0}$ is an increasing sequence which we call it *generalized ascending central series*. Moreover, $Z_n(P)$ is a subpolygroup of P, for every $n \geq 0$.

Proposition 6.11. [8, 17] *If $\langle P, \cdot, e, ^{-1} \rangle$ is a polygroup and $n \geq 0$, then*

(1) *$Z_n(P)$ is a complete subpolygroup of P;*
(2) *$g \cdot g^{-1} \subseteq Z_n(P)$, for every $g \in P$;*

(3) *$Z_n(P)$ is a normal subpolygroup of P.*

Theorem 6.12. [8, 17] *Let $\langle P, \cdot, e, ^{-1} \rangle$ be a polygroup. Then, P is nilpotent if and only if there exists $r \geq 0$ such that $Z_r(P) = P$.*

Let P_1, P_2 be two polygroups. Then, $P_1 \times P_2$ is nilpotent if and only if P_1 and P_2 are nilpotent. Let $\langle P_1, \cdot, e_1, ^{-1} \rangle$ and $\langle P_2, \circ, e_2, ^{-I} \rangle$ be two polygroups, and $\phi : P_1 \longrightarrow P_2$ be a good homomorphism. If ϕ is one to one and K_1 is a nilpotent subpolygroup of P_1, then $\phi(K_1)$ is a nilpotent subpolygroup of P_2.

Theorem 6.13. [8, 17]

- *Every commutative polygroup is solvable of length* 1*.*
- *Let P be a polygroup. Then, P is solvable if and only if $G = P/\beta^*$ is solvable.*
- *Every nilpotent polygroup is solvable.*
- *Every proper polygroup of order less than* 61 *is solvable.*

7. Strongly regular relations in ordered semihypergroups

In [16], Heidari and Davvaz studied a semihypergroup $(H, \circ)$ besides a binary relation $\leq$, where $\leq$ is a partial order relation such that satisfies the monotone condition. An *ordered semihypergroup* $(H, \circ, \leq)$ is a semihypergroup $(H, \circ)$ together with a partial order $\leq$ that is *compatible* with the hyperoperation, meaning that for any x, y, z in S,

$$x \leq y \Rightarrow z \circ x \leq z \circ y \text{ and } x \circ z \leq y \circ z.$$

Here, $z \circ x \leq z \circ y$ means for any $a \in z \circ x$ there exists $b \in z \circ y$ such that $a \leq b$. The case $x \circ z \leq y \circ z$ is defined similarly.

Let $(H, \circ, \leq_S)$ and $(T, \diamond, \leq_T)$ be two ordered semihypergroups. A mapping $f : S \rightarrow T$ is called a *homomorphism* if it satisfies the following conditions :

(1) $f(x \circ y) = f(x) \diamond f(y)$, for all $x, y \in H$;
(2) $x, y \in H$ and $x \leq_S y$ implies $f(x) \leq_T f(y)$.

Now, the following question is natural. Is there a strongly regular relation ρ on H for which H/ρ is an order semigroup The notion of pseudoorder on an ordered semigroup $(S, \cdot, \leq)$ was introduced and studied by Kehayopulu and Tsingelis [18]. Now, we define a similar concept for ordered semihypergroups. Let $(H, \circ, \leq)$ be an ordered semihypergroup. A relation ρ on S is called *pseudoorder* if

(1) $\leq \subseteq \rho$;
(2) $a\rho b$ and $b\rho c$ imply $a\rho c$;
(3) $a\rho b$ implies $a \circ c \overline{\overline{\rho}} b \circ c$ and $c \circ a \overline{\overline{\rho}} c \circ b$, for all $c \in H$.

Theorem 7.1. [11] *Let $(H, \circ, \leq)$ be an ordered semihypergroup and ρ be a pseudo-order on H. Then, there exists a strongly regular relation ρ^* on H such that H/ρ^* is an ordered semigroup.*

Sketch of proof. Suppose that ρ^* is the relation on H defined as follows :

$$\rho^* = \{(a, b) \in H \times H \mid a\rho b \text{ and } b\rho a\},$$

and define a relation $\preceq$ on H/ρ^* as follows :

$$\preceq := \{(\rho^*(x), \rho^*(y)) \in H/\rho^* \times H/\rho^* \mid \exists a \in \rho^*(x), \exists b \in \rho^*(y) \text{ such that } (a,b) \in \rho\}.$$

Note that we can show that

$$\rho^*(x) \preceq \rho^*(y) \Leftrightarrow x\rho y.$$

Theorem 7.2. [11] *Let $(H, \circ, \leq)$ be an ordered semihypergroup and ρ be a pseudoorder on H. Let*

$$\mathcal{X} = \{\theta \mid \theta \text{ is pseudoorder on } H \text{ such that } \rho \subseteq \theta\}.$$

Let $\mathcal{Y}$ be the set of all pseudoorders on H/ρ^*. Then, card($\mathcal{X}$)=card($\mathcal{Y}$).

Let $(H, \circ, \leq_S)$ be an ordered semihypergroup, ρ, θ be pseudoorders on H such that $\rho \subseteq \theta$. We define a relation θ/ρ on H/ρ^* as follows :

$$\theta/\rho := \{(\rho^*(a), \rho^*(b)) \in H/\rho^* \times H/\rho^* \mid \exists x \in \rho^*(a), \exists y \in \rho^*(b) \text{ such that } (x,y) \in \theta\}.$$

Then, we can see that

$$(\rho^*(x), \rho^*(b)) \in \theta/\rho \Leftrightarrow (a,b) \in \theta.$$

Theorem 7.3. [11] *Let $(H, \circ, \leq_S)$ be an ordered semihypergroup, ρ, θ be pseudoorders on H such that $\rho \subseteq \theta$. Then,*

(1) *θ/ρ is a pseudoorder on H/ρ^*.*
(2) *$(H/\rho^*)/(\theta/\rho)^* \cong H/\theta^*$.*

Let $(H, \circ, \leq_H)$ and $(T, \diamond, \leq_T)$ be two ordered semihypergroups, ρ_1, ρ_2 be two pseudoorders on H, T, respectively, and the map $f : S \to T$ be a homomorphism. Then, f is called a (ρ_1, ρ_2)-*homomorphism* if

$$(a,b) \in \rho_1 \Rightarrow (f(x), f(y)) \in \rho_2.$$

Let $(H, \circ, \leq_H)$ and $(T, \diamond, \leq_T)$ be two ordered semihypergroups, ρ_1, ρ_2 be two pseudoorders on H, T, respectively, and the map $f : H \to T$ be a (ρ_1, ρ_2)-homomorphism. Then, the map $\overline{f} : S/\rho_1^* \to T/\rho_2^*$ defined by

$$\overline{f}(\rho_1^*)) = \rho_2^*(f(x)), \text{ for all } x \in S$$

is a homomorphism of semigroups.

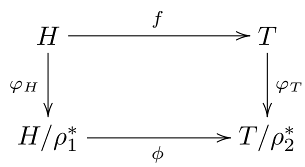
Theorem 7.4. [11] *Let $(H, \circ, \leq_H)$ and $(T, \diamond, \leq_T)$ be two ordered semihypergroups, ρ_1, ρ_2 be two pseudoorders on H, T, respectively, and the map $f : H \to T$ be a (ρ_1, ρ_2)-homomorphism. Then, the relation ρ_f defined by*

$$\rho_f := \{(\rho_1^*(x), \rho_1^*(y)) \mid \rho_2^*(f(x)) \preceq_T \rho_2^*(f(y))\}$$

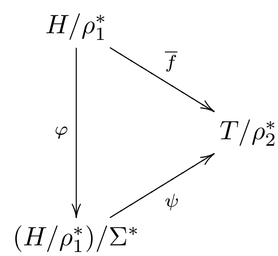
is a pseudoorder on H/ρ_1^*.

It is easy to see that $ker\overline{f} = \rho_f^*$.

Corollary 7.5. [11] *Let $(H, \circ, \leq_H)$ and $(T, \diamond, \leq_T)$ be two ordered semihypergroups, ρ_1, ρ_2 be two pseudoorders on H, T, respectively, and the map $f : H \to T$ be a (ρ_1, ρ_2)-homomorphism. Then, the following diagram is commutative.*



Theorem 7.6. [11] *Let $(H, \circ, \leq_H)$ and $(T, \diamond, \leq_T)$ be two ordered semihypergroups, ρ_1, ρ_2 be two pseudoorders on H, T, respectively, and the map $f : H \to T$ be a (ρ_1, ρ_2)-homomorphism. If Σ is a pseudoorder on H/ρ_1^* such that $\Sigma \subseteq (\rho_1)_f$. Then, the mapping $\psi : (H/\rho_1^*)/\Sigma^* \to T/\rho_2^*$ defined by $\psi(\Sigma^*(\rho_1^*(x))) = \overline{f}(\rho_1^*(a))$ is the unique homomorphism of $(H/\rho_1^*)/\Sigma^*$ into T/ρ_2^* such that the following diagram is commutative.*



Conversely, if Σ is a pseudoorder on H for which there exists a homomorphism $\psi : (H/\rho_1^*)/\Sigma^* \to T/\rho_2^*$ such that the above diagram commutes, then $\Sigma \subseteq (\rho_1)_f$.

Open problem 2. Is there a regular relation ρ on an ordered semihypergroup $(H, \circ, \leq_H)$ for which H/ρ is an ordered semihypergroup

8. Fundamental relations in hyperrings and H_v-rings

We present here the fundamental relation in the context of hyperrings. A multivalued system $(R, +, \cdot)$ is a (*general*) *hyperring* if (1) $(R, +)$ is a hypergroup; (2) $(R, \cdot)$ is a semihypergroup; (3) $(\cdot)$ is (strong) distributive with respect to $(+)$, i.e., for all x, y, z in R we have $x \cdot (y + z) = x \cdot y + x \cdot z$ and $(x + y) \cdot z = x \cdot z + y \cdot z$. A hyperring may be commutative with respect to $(+)$ or $(\cdot)$. If R is commutative with respect to both $(+)$ and $(\cdot)$, then we call it a commutative hyperring. The above definition contains the class of multiplicative hyperrings and additive hyperrings as well. In a hyperring, Vougiouklis [24] introduced the equivalence relation γ^*, which is similar to the relation β^*, defined in every hypergroup. Let $(R, +, \cdot)$ be a hyperring. We define the relation γ as follows :

$$a\gamma b \;\Leftrightarrow\; \{a, b\} \subseteq u, \text{ where } u \text{ is a finite sum of finite products of elements of } R.$$

We denote the transitive closure of γ by γ^*. Let $\mathcal{U}$ be the set of all finite sums of products of elements of R. We can rewrite the definition of γ^* on R as follows :

$$a\gamma^*b \;\Leftrightarrow\; \exists z_1,\ldots,z_{n+1}\in R \text{ with } z_1{=}a, z_{n+1}{=}b \text{ and } u_1,\ldots,u_n\in\mathcal{U}$$
$$\text{such that } \{z_i,z_{i+1}\}\subseteq u_i \text{ for } i\in\{1,\ldots,n\}.$$

Let $(R,+,\cdot)$ be a hyperring. Then, the relation γ^* is the smallest equivalence relation in R such that the quotient R/γ^* is a ring [24]. R/γ^* is called the *fundamental ring.* If $u=\sum_{j\in J}\left(\prod_{i\in I_j}x_i\right)\in\mathcal{U}$, then for all $z\in u$,

$$\gamma^*(u)=\oplus\sum_{j\in J}\Big(\odot\prod_{i\in I_j}\gamma^*(x_i)\Big)=\gamma^*(z),$$

where $\oplus\sum$ and $\odot\prod$ denote the sum and the product of classes. A multivalued system $(R,+,\cdot)$ is an *H_v-ring* [8] if (1) $(R,+)$ is an H_v-ring; (2) $(R,\cdot)$ is an H_v-semigroup; (3) $(\cdot)$ is (weak) distributive with respect to $(+)$, i.e., for all x,y,z in R we have $x\cdot(y+z)\cap x\cdot y+x\cdot z\neq\emptyset$ and $(x+y)\cdot z\cap x\cdot z+y\cdot z\neq\emptyset$. In what follows, we focus our attention on the β^* and γ^* relations defined on H_v-rings. Notice that two kinds of β^* relations can be defined on H_v-rings. We denote them by β^*_+ and $\beta^*_\cdot$. They are β^* relations with respect to addition and multiplication, respectively. If $(R,+,\cdot)$ is an H_v-ring, then the relations β_+ and $\beta_\cdot$ are defined as follows :

$$x\beta_+y \;\Leftrightarrow\; \exists z_1,\ldots,z_n{\in}R \text{ such that } \{x,y\}{\subseteq}z_1+\ldots+z_n,$$
$$x\beta_\cdot y \;\Leftrightarrow\; \exists z_1,\ldots,z_n{\in}R \text{ such that } \{x,y\}{\subseteq}z_1\cdot\ldots\cdot z_n.$$

β^*_+ and $\beta^*_\cdot$ are the transitive closures of the relations β_+ and $\beta_\cdot$.

Let $(R,+,\cdot)$ be an H_v-ring. We define γ^* as the smallest equivalence relation such that the quotient R/γ^* is a ring. γ^* is called the *fundamental equivalence relation* and R/γ^* is called the *fundamental ring.* Again, we denote the set of all finite polynomials of elements of R over $\mathbb{N}$ by $\mathcal{U}$. We define the relation γ as follows :

$$x\gamma y \;\Leftrightarrow\; \{x,y\}\subseteq u, \text{ where } u\in\mathcal{U}.$$

Theorem 8.1. [22, 24] *The fundamental equivalence relation γ^* is the transitive closure of the relation γ.*

In a multiplicative H_v-ring, the addition is an operation, while in an additive H_v-ring, the multiplication is an operation.

Theorem 8.2. [22, 24] *Let $(R,+,\cdot)$ be an H_v-ring. Then, $R/\gamma^*\cong(R/\beta^*_\cdot)/\beta^*_\uplus$, where $\beta^*_\uplus$ is the fundamental relation defined in $(R/\beta^*_\cdot,\uplus)$ by setting $\beta^*_\cdot(a)\uplus\beta^*_\cdot(b)=\{\beta^*_\cdot(c)\mid c\in\beta^*_\cdot(a)+\beta^*_\cdot(b)\}$.*

9. α^*-relations and fundamental commutative ring

Let $(R,+,\cdot)$ be a hyperring. The both $\oplus$ and $\odot$ on R/γ^* are defined as follows :

$$\gamma^*(a)\oplus\gamma^*(b)=\gamma^*(c), \text{ for all } c\in\gamma^*(a)+\gamma^*(b),$$
$$\gamma^*(a)\odot\gamma^*(b)=\gamma^*(d), \text{ for all } d\in\gamma^*(a)\cdot\gamma^*(b).$$

The commutativity in addition in rings can be related with the existence of the unit in multiplication. If e is the unit in a ring then for all elements a, b we have

$$(a+b)(e+e) = (a+b)e + (a+b)e = a+b+a+b,$$
$$(a+b)(e+e) = a(e+e) + b(e+e) = a+a+b+b.$$

So, $a+b+a+b = a+a+b+b$ gives $b+a = a+b$. Therefore, when we say $(R, +, \cdot)$ is a hyperring, $(+)$ is not commutative and there is not unit in the multiplication. So, the commutativity, as well as the existence of the unit, it is not assumed in the fundamental ring. Of course, we know there exist many rings (+ is commutative) while don't have unit. In [13], Davvaz and Vougiouklis defined a new fundamental relation to obtain an ordinary commutative ring from a hyperring. They introduced the following definition. If R is a hyperring, then we set $\alpha_0 = \{(x,x) \mid x \in R\}$ and, for every integer $n \geq 1$, α_n is the relation defined as follows :

$x\alpha_n y \;\Leftrightarrow\; \exists (k_1, k_2, \ldots, k_n) \in \mathbb{N}^n,\ \exists \sigma \in \mathbb{S}_n$ and $[\exists (x_{i1}, \ldots, x_{ik_i}) \in R^{k_i},\ \exists \sigma_i \in \mathbb{S}_{k_i}$, $(i = 1, \ldots, n)]$ such that

$$x \in \sum_{i=1}^{n} \Big(\prod_{j=1}^{k_i} x_{ij} \Big) \text{ and } y \in \sum_{i=1}^{n} A_{\sigma(i)},$$

where $A_i = \prod_{j=1}^{k_i} x_{i\sigma_i(j)}$. Obviously, for every $n \geq 1$, the relation α_n is symmetric, and the relation $\alpha = \bigcup_{n \geq 0} \alpha_n$ is reflexive and symmetric. Let α^* be the transitive closure of α. Then,

Proposition 9.1. [13] *α^* is a strongly regular relation both on $(R, +)$ and $(R, \cdot)$.*

Theorem 9.2. [13] *The quotient R/α^* is a commutative ring.*

Theorem 9.3. [13] *The relation α^* is the smallest equivalence relation such that the quotient R/α^* is a commutative ring.*

Let $(R, +, \cdot)$ be a hyperring, then we define the relations γ and γ_+ on R as follows :

$$x\gamma y \;\Leftrightarrow\; \exists n \in \mathbb{N},\ \exists (z_1, \ldots, z_n) \in R^n, \exists \sigma \in \mathbb{S}_n :\ x \in \prod_{i=1}^{n} z_i,\quad y \in \prod_{i=1}^{n} z_{\sigma(i)},$$

$$x\gamma_+ y \;\Leftrightarrow\; \exists m \in \mathbb{N},\ \exists (y_1, \ldots, y_m) \in R^m, \exists \tau \in \mathbb{S}_m :\ x \in \sum_{i=1}^{m} y_i,\quad y \in \sum_{i=1}^{m} y_{\tau(i)}.$$

We denote γ^* and γ_+^* as the transitive closure of the relations γ and γ_+, respectively. We have $\gamma^* \cup \gamma_+^* \subseteq \alpha^*$.

Theorem 9.4. [13] *For all additive hyperrings we have $\alpha^* = \gamma_+^*$.*

Let H be a set and $\circ : H \times H \to \mathcal{P}^*(H)$ be a hyperoperation. The $(\circ)$ is called *weak commutative*, we write COW, if

$$x \circ y \cap y \circ x \neq \emptyset, \text{ for all } x, y \in H.$$

A hyperring $(R, +, \cdot)$ can be COW with respect to $(+)$ or $(\cdot)$. If it is in both COW, we call it COW hyperring.

Theorem 9.5. [13] *If R is a COW hyperring, then $\alpha^* = \gamma^*$.*

Let R is a Krasner hyperring. If R is commutative or have a unit, then $\gamma^* = \alpha^*$.

Proposition 9.6. [20] *If R is a hyperring and $n \geq 1$, then $\alpha_n \subseteq \alpha_{n+1}$.*

Lemma 9.7. [20] *If $x\alpha_n y$, then for every $a \in R$, $x + a \ \overline{\overline{\alpha_{n+1}}}\ y + a$ and $xa \ \overline{\overline{\alpha_n}}\ ya$.*

Let M be a non-empty subset of R. We say that M is an *α-part* if for every $n \in \mathbb{N}$, $i = 1, 2, \ldots, n$, $\forall k_i \in \mathbb{N}$, $\forall (z_{i1}, z_{i2}, \ldots, z_{ik_i}) \in R^{n_i}$, $\forall \sigma \in \mathbb{S}_n, \forall \sigma_i \in \mathbb{S}_{k_i}$, we have

$$\sum_{i=1}^{n} \Big(\prod_{j=1}^{k_i} z_{ij} \Big) \cap M \neq \emptyset \Rightarrow \sum_{i=1}^{n} A_{\sigma(i)} \subseteq M,$$

where $A_i = \prod_{j=1}^{k_i} z_{i\sigma_i(j)}$ [20].

Proposition 9.8. [20] *Let M be a non-empty subset of a hyperring R. The following conditions are equivalent :*

(1) *M is a α-part of R;*
(2) *$x \in M, x \ \alpha \ y \Rightarrow y \in M$;*
(3) *$x \in M, x \ \alpha^* \ y \Rightarrow y \in M$.*

For every element x of a hyperrring R, set :

$[x]^n_{k_1,k_2,\ldots,k_n} = \{(x_{i1}, x_{i2}, \ldots, x_{ik_i}) \in R^{k_i} | i = 1, 2 \ldots, n\}$;

$T_n(x) = \Big\{ [x]^n_{k_1,k_2,\ldots,k_n} \mid x \in \sum_{i=1}^{n} (\prod_{j=1}^{k_i} x_{ij}) \Big\}$;

$P_n(x) = \bigcup \Big\{ \sum_{i=1}^{n} (\prod_{j=1}^{k_i} x_{\sigma(i)\sigma_{\sigma(i)}(j)}) \mid \sigma \in \mathbb{S}_n, \sigma_i \in \mathbb{S}_{k_i}, [x]^n_{k_1,k_2,\ldots,k_n} \in T_n(x) \Big\}$;

$P(x) = \bigcup_{n \geq 1} P_n(x)$.

For every $x \in R$, $P(x) = \{y \in R \mid x \ \alpha \ y\}$.

Theorem 9.9. [20] *Let R be a hyperring. The following conditions are equivalent :*

(1) *α is transitive;*
(2) *for every $x \in R, \alpha^*(x) = P(x)$;*
(3) *for every $x \in R$, $P(x)$ is a α-part of R.*

A hyperring R is said to be *n-complete* [20] if $\forall (k_1, \ldots, k_n) \in \mathbb{N}^n, \forall (x_{ij}, \ldots, x_{ik_i}) \in R^{k_i}$, then

$$\gamma\Big(\sum_{i=1}^{n} (\prod_{j=1}^{k_i} x_{ij}) \Big) = \sum_{i=1}^{n} \Big(\prod_{j=1}^{k_i} x_{ij} \Big).$$

A hyperring R is said to be *α_n-complete* if $\forall (k_1, \ldots, k_n) \in \mathbb{N}^n, \forall (x_{ij}, \ldots, x_{ik_i}) \in R^{k_i}, \forall \sigma \in \mathbb{S}_n, \forall \sigma_i \in \mathbb{S}_{k_i}, i = 1, \ldots, n$ then

$$\alpha\Big(\sum_{i=1}^{n} (\prod_{j=1}^{k_i} x_{ij}) \Big) = \sum_{i=1}^{n} A_{\sigma(i)},$$

where $A_i = \prod_{j=1}^{k_i} x_{i\sigma_i(j)}$. If R is a commutative hyperring then R is an α_n-complete hyperring if and only if R is an n-complete hyperring.

Proposition 9.10. [20] *A hyperring R is α_n-complete if and only if* $\forall (k_1, \ldots, k_n) \in \mathbb{N}^n$, $\forall (x_{ij}, \ldots, x_{ik_i}) \in R^{k_i}, \forall \sigma \in \mathbb{S}_n, \forall \sigma_i \in \mathbb{S}_{k_i}, i = 1, \ldots, n$ *and for every* $x \in \sum_{i=1}^{n} (\prod_{j=1}^{k_i} x_{ij})$, *we have*

$$\alpha(x) = \sum_{i=1}^{n} A_{\sigma(i)},$$

where $A_i = \prod_{j=1}^{k_i} x_{i\sigma_i(j)}$.

Proposition 9.11. [20] *If R is an α_n-complete hyperring then $\alpha = \alpha_n$.*

Proposition 9.12. [20] *If R is an α_n-complete hyperring then for all* $[(k_1, \ldots, k_n) \in \mathbb{N}^n, (x_{ij}, \ldots, x_{ik_i}) \in R^{k_i}, \sigma \in \mathbb{S}_n, \sigma_i \in \mathbb{S}_{k_i}]$, $\sum_{i=1}^{n} A_{\sigma(i)}$ *is an α-part of R.*

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