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H_v-LIE ALGEBRAS AND THEIR REPRESENTATIONS

T. VOUGIOUKLIS

Democritus University of Thrace School of Education, 68100
Alexandroupolis, Greece
tvougiou@eled.duth.gr

Abstract. On the general H_v-Lie algebras we investigate several classes mainly when only the bracket is a hyperoperation. In the finite dimensional Lie algebras and in the infinite dimensional Kac-Moody Lie algebras, as well, the matrix representation problem is introduced and studied. Finally we present the hyper-representation problem on non-square matrices using the helix-hopes. Examples are presented by using some general classes as the P-hopes and the ∂-hopes.

1. The H_v-structures

We deal with hyperstructures called H_v-structures introduced in 1990 [10], which satisfy the weak axioms where the non-empty intersection replaces the equality.
Basic definitions:
In a set H equipped with a hyperoperation (abbreviation *hyperoperation = hope)* $\cdot : H \times H \to P(H) - \{\varnothing\}$, we abbreviate by *WASS* the *weak associativity*: $(xy)z \cap x(yz) \neq \varnothing, \forall x, y, z \in H$ and by *COW* the *weak commutativity*: $xy \cap yx \neq \varnothing, \forall x, y \in H$. The hyperstructure $(H, \cdot)$ is called an H_v*-semigroup* if it is *WASS*, it is called H_v*-group* if it is reproductive H_v-semigroup. The hyperstructure $(R, +, \cdot)$ is called an H_v*-ring* if $(+)$ and $(\cdot)$ are WASS, the reproduction axiom is valid for $(+)$ and $(\cdot)$ is *weak distributive* with respect to $(+)$: $x(y+z) \cap (xy+xz) \neq \varnothing$, $(x+y)z \cap (xz+yz) \neq \varnothing$, $\forall x, y, z \in R$.

For more definitions and applications on H_v-structures, see books [1],[12].

The fundamental relations β^*, γ^* and ϵ^*, are defined, in H_v-groups, H_v-rings and H_v-vector space, respectively, as the smallest equivalences so that the quotient would be group, ring and vector space, respectively [10]. A way to find the fundamental classes is given by analogous theorems to the following [9], [10], [14], [15], [16]:

Theorem 1.1. *Let* $(\boldsymbol{H}, \cdot)$ *be an* H_v*-group and denote by* $\boldsymbol{U}$ *the set of all finite products of elements of H. We define the relation* β *in H by setting* $x\beta y$ *iff* $\{x, y\} \subset \boldsymbol{u}$ *where* $\boldsymbol{u} \in \boldsymbol{U}$*. Then the fundamental relation* β^* *is the transitive closure of* β*.*

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There are analogous theorems for the γ^* in H_v-rings and ϵ^* in H_v-vector spaces.

An element is called single [12], if its fundamental class is singleton.

The fundamental relations are used for general definitions. Thus, an H_v-ring $(R,+,\cdot)$ is called H_v-field if R/γ^* is a field. In the sequence the H_v-vector space is defined.

Definition 1.2. [12],[16] Let $(R,+,\cdot)$ be an H_v-ring, $(M,+)$ be COW H_v-group and there exists an external hope $\cdot : R\times M \to P(M) : (a,x) \to ax$, such that, $\forall a,b \in R$ and $\forall x,y \in M$ we have

$$a(x+y)\cap(ax+ay)\neq\emptyset,(a+b)x\cap(ax+bx)\neq\emptyset,(ab)x\cap a(bx)\neq\emptyset,$$

then M is called an H_v-module over R. In the case of an H_v-field F instead of H_v-ring R, then the H_v-*vector space* is defined.

The fundamental relation ϵ^* is defined to be the smallest equivalence such that the quotient M/ϵ^* is a module (respectively, a vector space) over the fundamental ring R/γ^* (resp. the fundamental field F/γ^*).

Theorem 1.3. *Let $(\boldsymbol{M},+)$ be an H_v-module over the H_v-ring $\boldsymbol{R}$. Denote by $\boldsymbol{U}$ the set of all expressions consisting of finite hopes either on $\boldsymbol{R}$ and $\boldsymbol{M}$ or the external hope applied on finite sets of elements $\boldsymbol{R}$ and $\boldsymbol{M}$. We define the relation ϵ in $\boldsymbol{M}$ as follows:*

$$x\epsilon y \textit{ iff } \{x,y\}\subset u \textit{ where } u\in \boldsymbol{U}$$

Then the relation ϵ^* is the transitive closuer of the relation ϵ

Proof. Let $\underline{\epsilon}$ be the transitive closure of ϵ, and denote by $\underline{\epsilon}(x)$ the class of the element x. First we prove that the quotient set $M/\underline{\epsilon}$ is a module over R/γ^*.

In $\mathbf{M}/\underline{\epsilon}$ the sum $(\oplus)$ and the external product $(\otimes)$,using the γ^* classes in $\mathbf{R}$, are defined in the usual manner:

$$\underline{\epsilon}(x)\oplus\underline{\epsilon}(y)=\{\underline{\epsilon}(z):z\in\underline{\epsilon}(x)+\underline{\epsilon}(y)\},$$

$$\gamma^*(a)\otimes\underline{\epsilon}(x)=\{\underline{\epsilon}(z):z\in\gamma^*(a)\cdot\underline{\epsilon}(x)\},\ \ \forall a\in\mathbf{R},\ x,y\in\mathbf{M}$$

Take $x'\in\underline{\epsilon}(x)$, $y'\in\underline{\epsilon}(y)$. Then we have $x'\underline{\epsilon}x$ iff $\exists x_1,...,x_{m+1}$ with $x_1=x', x_{m+1}=x$ and $u_1,...,u_m\in\mathbf{U}$ such that $\{x_i,x_{i+1}\}\subset u_i$, $i=1,...,m$, and $y'\underline{\epsilon}y$ iff $\exists y_1,...,y_{n+1}$ with $y_1=y', y_{n+1}=y$ and $v_1,...,v_n\in\mathbf{U}$ such that $\{y_j,y_{j+1}\}\subset v_j,\ i=1,...,n$. From the above we obtain

$$\{x_i,x_{i+1}\}+y_1\subset u_i+v_1,\ i=1,...,m-1,$$

$$x_{m+1}+\{y_j,y_{j+1}\}\subset u_m+v_j,\ j=1,...,n.$$

The sums

$$u_i+v_1=t_i,\ i=1,..,.m-1 \text{ and } u_m+v_j=t_{m+j-1},\ j=1,...,n$$

are also elements of $\mathbf{U}$, therefore $t_k\in\mathbf{U}$ for all $k\in\{1,...,m+n-1\}$. Now, pick up elements $z_1,...,z_{m+n}$ such that

$$z_i\in x_i+y_1,\ i=1,...,n \text{ and } z_{m+j}\in x_{m+1}+y_{j+1},\ j=1,...,n,$$

therefore, using the above relations we obtain $\{z_k, z_{k+1}\} \subset t_k,\ k = 1, ..., m+n-1$. Thus, every element $z_1 \in x_1 + y_1 = x' + y'$ is $\underline{\epsilon}$ equivalent to every element $z_{m+n} \in x_{m+1} + y_{n+1} = x + y$. Thus $\underline{\epsilon}(x) \oplus \underline{\epsilon}(y)$ is a singleton so we can write

$$\underline{\epsilon}(x) \oplus \underline{\epsilon}(y) = \underline{\epsilon}(z) \text{ for all } z \in \underline{\epsilon}(x) + \underline{\epsilon}(y)$$

In a similar way, using the properties of γ^* in $\mathbf{R}$, one can prove that

$$\gamma^*(a) \otimes \underline{\epsilon}(x) = \underline{\epsilon}(z) \text{ for all } z \in \gamma^*(a) \cdot \underline{\epsilon}(x)$$

The WASS and the weak distributivity on $\mathbf{R}$ and $\mathbf{M}$ guarantee that the associativity and the distributivity are valid for the quotient $\mathbf{M}/\underline{\epsilon}$ over $\mathbf{R}/\gamma^*$. Therefore $\mathbf{M}/\underline{\epsilon}$ is a module over R/γ^*.

Now let σ be an equivalence relation in $\mathbf{M}$ such that $\mathbf{M}/\sigma$ is a module over R/γ^*. Denote $\sigma(x)$ the class of x. Then $\sigma(x) \oplus \sigma(y)$ and $\gamma^*(a) \otimes \sigma(x)$ are singletons for all $a \in \mathbf{R}$ and $x, y \in \mathbf{M}$, i.e.

$$\sigma(x) \oplus \sigma(y) = \sigma(z) \text{ for all } z \in \sigma(x) + \sigma(y),$$

$$\gamma^*(a) \otimes \sigma(x) = \sigma(z) \text{ for all } z \in \gamma^*(a) \cdot \sigma(x).$$

Thus we can write, for every $a \in \mathbf{R}$, $x, y \in \mathbf{M}$ and $A \subset \gamma^*(a)$, $\mathbf{X} \subset \sigma(x)$,$\mathbf{Y} \subset \sigma(x)$

$$\sigma(x) \oplus \sigma(y) = \sigma(x + y) = \sigma(\mathbf{X} + \mathbf{Y}),\ \gamma^*(a) \otimes \sigma(x) = \sigma(ax) = \sigma(\mathbf{A} \cdot \mathbf{X})$$

By induction, we extend these relations on finite sums and products. Thus, for every $u \in \mathbf{U}$, we have $\sigma(x) = \sigma(u)$ for all $x \in u$. Consequently

$$x' \in \epsilon(x) \text{ implies } x' \in \sigma(x) \text{ for every } x \in \mathbf{M}.$$

But σ is transitively closed, so we obtain:

$$x' \in \underline{\epsilon}(x) \text{ implies } x' \in \sigma(x).$$

That means that $\underline{\epsilon}$ is the smallest equivalence relation in $\mathbf{M}$ such that $\mathbf{M}/\underline{\epsilon}$ is a module over $\mathbf{R}/\gamma^*$, i.e. $\underline{\epsilon} = \epsilon^*$. □

Let $(H, \cdot), (H, *)$ be H_v-semigroups defined on the same set "H". The hope $(\cdot)$ is called *smaller* than the hope $(*)$, and $(*)$ *greater* than $(\cdot)$, iff there exists an

$$f \in Aut(H, *) \text{ such that } xy \subset f(x * y),\ \forall x, y \in H.$$

Then we write $\cdot \leq *$ and we say that $(H, *)$ contains $(H, \cdot)$. If $(H, \cdot)$ is a structure then it is called *basic structure* and $(H, *)$ is called $H_b - structure$ [11], [13], [4].

Theorem 1.4. *(The Little Theorem). Greater hopes than the ones which are WASS or COW, are also WASS or COW, respectively.*

Definition 1.5. [14],[16] Let $(H, \cdot)$ be hypergroupoid. We *remove* $h \in H$, if we consider the restriction of $(\cdot)$ in the set $H - \{h\}$. $\underline{h} \in H$ *absorbs* $h \in H$ if we replace h by $\underline{h}$ and h does not appear in the structure. $\underline{h} \in H$ *merges* with $h \in H$, if we take as product of any $x \in H$ by $\underline{h}$, the union of the results of x with both h, $\underline{h}$, and consider h and $\underline{h}$ as one class with representative $\underline{h}$, therefore the element h does not appeared in the hyperstructure.

Theorem 1.6. *Let $(H, \cdot)$ be an H_v-group. If an element h absorbs all elements of its own fundamental class then this element becomes a single in the new H_v-group.*

Proof. Let $h \in \beta^*(h)$, then, by the definition of the "absorb", h is replaced by $\underline{h}$ that means that $\beta^*(\underline{h}) = \{\underline{h}\}$. Moreover, for all $x \in H$, the fundamental property of the product of classes

$$\beta^*(x) \cdot \beta^*(\underline{h}) = \beta^*(x\underline{h}) \text{ becomes } \beta^*(x) \cdot \underline{h} = \beta^*(x\underline{h}),$$

and from the reproductivity (see [12] page 19) we obtain $x \cdot \underline{h} = \beta^*(x\underline{h}), \forall x \in \beta^*(x)$. This is the basic property that enjoys any single element [12].

□

Definition 1.7. Let $(L,+)$ be an H_v-vector space over the H_v-field $(F,+,\cdot)$, $\phi : F \to F/\gamma^*$ the canonical map and $\omega_F = \{x \in F : \phi(x) = 0\}$, where 0 is the zero of the fundamental field F/γ^*. Similarly, let ω_L be the core of the canonical map $\phi' : L \to L/\epsilon^*$ and denote by the same symbol 0 the zero of L/ϵ^*. Consider the *bracket (commutator) hope*:

$$[,] : L \times L \to P(L) : (x,y) \to [x,y]$$

then **L** is an H_v-*Lie algebra* over F if the following axioms are satisfied:

(L1) The bracket hope is bilinear, i.e.
$[\lambda_1 x_1 + \lambda_2 x_2, y] \cap (\lambda_1[x_1,y] + \lambda_2[x_2,y]) \neq \varnothing$
$[x, \lambda_1 y_1 + \lambda_2 y_2] \cap (\lambda_1[x,y_1] + \lambda_2[x,y_2]) \neq \varnothing$,
$\forall x, x_1, x_2, y, y_1, y_2 \in L, \lambda_1, \lambda_2 \in F$

(L2) $[x,x] \cap \omega_L \neq \varnothing, \quad \forall x \in L$

(L3) $([x,[y,z]] + [y,[z,x]] + [z,[x,y]]) \cap \omega_L \neq \varnothing, \quad \forall x,y,z \in L$

A class of H_v-structures, introduced in [12] is the following:

An H_v-structure is called *very thin* iff all hopes are operations except one, which has all hyperproducts singletons except only one, which is a subset of cardinality more than one. Therefore, in a very thin H_v-structure in a set H there exists a hope $(\cdot)$ and a pair $(a,b) \in H^2$ for which $ab = A$, with $cardA > 1$, and all the other products, with respect to any other hopes (so they are operations), are singletons.

2. Some general classes of H_v-structures

A well known and large class of hopes is given as follows [8],[12]:

Let $(G,\cdot)$ be a groupoid then for every $P \subset G$, $P \neq \varnothing$, we define the following hopes called P-*hopes*: for all $x, y \in G$

$$\underline{P} : x\underline{P}y = (xP)y \cup x(Py),$$

$$\underline{P}_r : x\underline{P}_r y = (xy)P \cup x(yP), \quad \underline{P}_l : x\underline{P}_l y = (Px)y \cup P(xy).$$

The $(G,\underline{P})$,$(G,\underline{P}_r)$ and $(G,\underline{P}_l)$ are called P-*hyperstructures*. The most usual case is if $(G,\cdot)$ is semigroup, then $x\underline{P}y = (xP)y \cup x(Py) = xPy$ and $(G,\underline{P})$ is a semihypergroup but we do not know about $(G,\underline{P}_r)$ and $(G,\underline{P}_l)$. In some cases, depending on the choice of P, the $(G,\underline{P}_r)$ and $(G,\underline{P}_l)$ can be associative or WASS.

A generalization of P-hopes is the following [2]:

Construction 2.1. Let $(G,\cdot)$ be an abelian group and P any subset of G with more than one elements. We define the hope $\times_p$ as follows:

$$x \times_p y = \begin{cases} x\cdot P\cdot y = \{x\cdot h\cdot y | h\in P\} & \text{if } x\neq e \text{ and } y\neq e \\ x\cdot y & \text{if } x=e \text{ or } y=e \end{cases}$$

we call this hope P_e-hope. The hyperstructure $(G,\times_p)$ is an abelian H_v-group.

A general definition of hopes, is the following:

Definition 2.2. Let H be a set with "n" operations (or hopes) $\otimes_1, \otimes_2, ..., \otimes_n$ and one map (or multivalued map) $f: H\to H$, then n hopes $\partial_1, \partial_2, ..., \partial_n$ on H are defined, called *∂-hopes* by putting

$$x\partial_i y = \{f(x)\otimes_i y, x\otimes_i f(y)\}, \forall x,y\in H, i\in\{1,2,...,n\}$$

or in case where $\otimes_i$ is hope or "f" is multivalued map we have

$$x\partial_i y = (f(x)\otimes_i y)\cup(x\otimes_i f(y)), \forall x,y\in H, i\in\{1,2,...,n\}$$

One can see that if $\otimes_i$ is associative then ∂_i is WASS.

Remark that one can use several maps "f", instead of only one, in an analogous way. We can define *∂-hopes* on the union of maps:

Definition 2.3. Let $(G,\cdot)$ groupoid and $f_i: G\to G, i\in I$, set of maps on G. Take the map $f_\cup: G\to \mathbf{P}(G)$ such that $f_\cup(x) = \{f_i(x)|i\in I\}$ and we call it the union of the $f_i(x)$. We call the *union ∂-hope* (∂), on G if we consider the map $f_\cup(x)$. An important case for a map f, is to take the union of this with the identity id. Thus, we consider the map $\underline{f}\equiv f_\cup(id)$, so $\underline{f}(x)=\{x,f(x)\}, \forall x\in G$, which is called $b-\partial-hope$, we denote it by $(\underline{\partial})$, so we have

$$x\underline{\partial}y = \{xy, f(x)\cdot y, x\cdot f(y)\}, \forall x,y\in G.$$

Remark that $\underline{\partial}$ contains the operation $(\cdot)$, so it is b-operation. Moreover, if $f: G\to \mathbf{P}(G)$ is multivalued then the b-∂-hopes is defined by using the

$$\underline{f}(x) = \{x\}\cup f(x), \forall x\in G.$$

Motivation for the definition of ∂-hope is the derivative where only multiplication of functions can be used. Therefore, for functions s(x), t(x), we have $s\partial t = \{s't, st'\}, (')$ is the derivative.

Example. Application on derivative: consider all polynomials of first degree $g_i(x) = a_i x + b_i$. We have $g_1\partial g_2 = \{a_1a_2x + a_1b_2, a_1a_2x + b_1a_2\}$, so it is a hope in the set of first degree polynomials. Moreover all polynomials $x+c$, where c be a constant, are units.

Properties 2.4. [17] Let $(G,\cdot)$ semigroup then: (a) $\forall f: G\to G$, the hope (∂) is WASS. (b) If "f" is homomorphism then (∂) remains WASS. (c) If "f" is homomorphism and projection, i.e. $f^2=f$, then (∂) is associative.

Properties 2.5. [17] If $(G,\cdot)$ is a semigroup then, for every $f: G\to G$, the b-∂-hope $(\underline{\partial})$ is WASS.

Properties in the general case where $(G,\cdot)$ be a groupoid and $f:G\to G$ be any map:

Properties 2.6. [17] *Reproductivity.* For the reproductivity we must have $x\partial G=\cup_{g\in G}\{f(x)\cdot g, x\cdot f(g)\}=G$ and $G\partial x=\cup_{g\in G}\{f(g)\cdot x, g\cdot f(x)\}=G$. So if $(\cdot)$ is reproductive then (∂) is reproductive, since $\cup_{g\in G}\{f(x)\cdot g\}=G$ and $\cup_{g\in G}\{g(x)\cdot f\}=G$.

Commutativity. If $(\cdot)$ is commutative then (∂) is commutative. If "f" is into the centre of G, then (∂) is commutative. If $(\cdot)$ is COW then, (∂) is COW.

Unit elements. u is a right unit element if $x\partial u=\{f(x)\cdot u, x\cdot f(u)\}\ni x$. So $f(u)=e$, where e be a unit in $(G,\cdot)$. The elements of the kernel of f, are the units of (G,∂).

Inverse elements. Let $(G,\cdot)$ is a monoid with unit e and u be a unit in (G,∂), then $f(u)=e$. For given x, the element x' is an inverse with respect to u, if

$$x\partial x'=\{f(x)\cdot x', x\cdot f(x')\}\ni u \text{ and } x'\partial x=\{f(x')\cdot x, x'\cdot f(x)\}\ni u.$$

So, $x'=(f(x))^{-1}u$ and $x'=u(f(x))^{-1}$, are the right and left inverses, respectively. We have two-sided inverses iff $f(x)u=uf(x)$.

Similar properties for multivalued maps, are obtained. For proofs of the following see [17].

Proposition 2.7. *Let $(G,\cdot)$ be group then, $\forall f:G\to G, the (G,\partial)$ is H_v-group.*

Proposition 2.8. *Let $(G,\cdot)$ be a group and $f(x)=a$, constant map on G. Then $(G,\partial)/\beta^*$ is singleton. If $f(x)=e$, then we obtain $x\partial y=\{x,y\}$, the smallest incidence hope.*

We define ∂-hope on rings and to other more complicate structures.

Definition 2.9. Let $(R,+,\cdot)$ be ring and $f:R\to R$, $g:R\to R$ be two maps. We define two hopes (∂_+) and $(\partial.)$, called both *∂-hopes*, on R as follows

$$x\partial_+ y=\{f(x)+y, x+f(y)\} \text{ and } x\partial.y=\{g(x)\cdot y, x\cdot g(y)\}, \forall x,y\in G.$$

A hyperstructure $(R,+,\cdot)$, where $(+)$, $(\cdot)$ be two hopes which satisfy all H_v-ring axioms, except the weak distributivity, will be called an *H_v-near-ring.*

Proposition 2.10. *Let $(R,+,\cdot)$ ring and $f:R\to R$, $g:R\to R$ maps. The hyperstructure $(R,\partial_+,\partial.)$, called theta, is an Hv-near-ring. Moreover $(+)$ is commutative.*

More properties are valid if we replace (∂) by the corresponding *b-∂-hopes* $(\underline{\partial})$.

Proposition 2.11. *Let $(R,+,\cdot)$ ring and $f:R\to R, g:R\to R$ maps. The $(R,\partial_+,\partial.)$, is an H_v-ring.*

Properties 2.12. (*Special classes*). The theta hyperstructure $(R,\partial_+,\partial.)$ takes a new form and have some properties in several forms as the following ones:

(a) If $f(x)\equiv g(x), \forall x\in R$, i.e. the two maps coincide, $(R,\partial_+,\partial.)$ is an H_v-ring.

(b) If $g(x) = x, \forall x \in R$, i.e. only the f in addition is used, then we have

$$x(y\partial_+ z) = \{xf(y) + xz, xy + xf(z)\}, (xy)\partial_+(xz) = \{f(xy) + xz, xy + f(xz)\}$$

Therefore, $x(y\partial_+ z) \cap (xy)\partial_+(xz) = \varnothing$.

(c) If $f(x) = x, \forall x \in R$, then $(R, +, \partial.)$ becomes a multiplicative H_v-ring.

Now we can see ∂-hopes in H_v-vector spaces and H_v-Lie algebras:

Theorem 2.13. *Let $(\mathbf{V}, +, \cdot)$ be an algebra over the field $(F, +, \cdot)$ and $f : \mathbf{V} \to \mathbf{V}$ be map. Consider the ∂-hope defined only on the multiplication of the vectors $(\cdot)$, then $(\mathbf{V}, +, \partial)$ is an H_v-algebra over F, where the related properties are weak. If f is linear then we have strong properties.*

Theorem 2.14. *Let $(A, +, \cdot)$ be an algebra over the field F. Take any map $f : A \to A$, then the ∂-hope on the Lie bracket $[x, y] = xy - yx$, is defined by*

$$x\partial y = \{f(x)y - f(y)x, f(x)y - yf(x), xf(y) - f(y)x, xf(y) - yf(x)\}.$$

then $(A, +, \partial)$ is an H_v-algebra over F, with respect to the ∂-hopes on Lie bracket, where the weak anti-commutativity and the inclusion linearity is valid.

3. Representations

H_v-structures are used in Representation Theory (abbreviate by *rep*) Theory of H_v-groups which can be considered either by generalized permutations or by H_v-matrices [11],[12]. Reps by generalized permutations can be acheived by using left or right translations.

The rep problem by H_v-matrices is the following:

H_v-matrix is called a matrix if has entries from an H_v-ring. The hyperproduct of H_v-matrices **A**$= (a_{ij})$ and **B**$= (b_{ij})$, of type $m \times n$ and $n \times r$, respectively, is a set of $m \times r$ H_v-matrices, defined in a usual manner:

$$A \cdot B = (a_{ij}) \cdot (b_{ij}) = \{C = (c_{ij}) | (c_{ij}) \in \oplus \sum a_{ik} \cdot b_{kj}\},$$

where $(\oplus)$ denotes the *n-ary circle hope* on the hyperaddition.

Definition 3.1. Let $(H, \cdot)$ be an H_v-group,$(R, +, \cdot)$ be an H_v-ring R and consider a set $M_R = \{(a_{ij}) | a_{ij} \in R\}$ then any map

$$T : H \to M_R : h \mapsto T(h) \text{ with } T(h_1 h_2) \cap T(h_1)T(h_2) \neq \varnothing, \forall h_1, h_2 \in H.$$

is called H_v-matrix rep. If $T(h_1 h_2) \subset T(h_1)T(h_2)$, then **T** is an *inclusion rep*, if $T(h_1 h_2) = T(h_1)T(h_2)$, then **T** is a *good rep*, then an induced rep T^* for the hypergroup algebra is obtained. If T is 1:1 and good then it is *faithful rep*.

Using several classes of H_v-structures one can face several reps [12]:

Definition 3.2. Let $\mathbf{M} = \mathbf{M}_{m\times n}$ be a module of $m \times n$ matrices over a ring **R** and $\mathbf{P} = \{P_i : i \in I\} \subseteq \mathbf{M}$. We define, a kind of, a P-hope $\underline{P}$ on **M** as follows

$$\underline{P} : \mathbf{M} \times \mathbf{M} \to \mathbf{P}(\mathbf{M}) : (A, B) \to A\underline{P}B = \{AP_i^t B : i \in I\} \subseteq \mathbf{M}$$

where P^t denotes the transpose of the matrix P.

The hope $\underline{P}$, which is a bilinear map, is a generalization of Rees' operation where, instead of one sandwich matrix, a set of sandwich matrices is used. The hope P is strong associative and the inclusion distributivity with respect to addition of matrices

$$A\underline{P}(B+C) \subseteq A\underline{P}B + A\underline{P}C, \forall A, B, C \in \mathbf{M}$$

is valid. Therefore, $(\mathbf{M}, +, \underline{P})$ defines a multiplicative hyperring on non-square matrices

Definition 3.3. Let $\mathbf{M} = \mathbf{M}_{m\times n}$ be a module of $m \times n$ matrices over R and let us take sets

$$\mathbf{S} = \{s_k : k \in K\} \subseteq R, \quad \mathbf{Q} = \{Q_j : j \in J\} \subseteq \mathbf{M}, \quad \mathbf{P} = \{P_i : i \in I\} \subseteq \mathbf{M}.$$

Define three hopes as follows

$$\underline{S} : R \times \mathbf{M} \to \mathbf{P}(\mathbf{M}) : (r, A) \to r\underline{S}A = \{(rs_k)A : k \in K\} \subseteq \mathbf{M}$$

$$\underline{Q}_+ : \mathbf{M} \times \mathbf{M} \to \mathbf{P}(\mathbf{M}) : (A, B) \to A\underline{Q}_+B = \{A + Q_j + B : j \in J\} \subseteq \mathbf{M}$$

$$\underline{P} : \mathbf{M} \times \mathbf{M} \to \mathbf{P}(\mathbf{M}) : (A, B) \to A\underline{P}B = \{AP_i^t B : i \in I\} \subseteq \mathbf{M}$$

Then $(\mathbf{M}, \underline{S}, \underline{Q}_+, \underline{P})$ is a hyperalgebra over $\mathbf{R}$ called *general matrix P-hyperalgebra.*

Remark. In a similar way a generalization of this hyperalgebra can be defined if one consider an H_v-ring or an H_v-field instead of a ring and using H_v-matrices instead of matrices.

Definition 3.4. Let $A = (a_{ij}), B = (b_{ij}) \in M_{m\times n}$, we call (A,B) a *unitize pair* of matrices if $A^t B = I_n$, where I_n denotes the $n \times n$ unit matrix.

We prove the following theorem which can be applied in the classical theory.

Theorem 3.5. *In the above notation if $m < n$, then there is no unitize pair.*

Proof. Suppose that $A^t B = (c_{ij})$, that is $c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$, and we denote by A_m the block of the matrix A such that $A_m = (a_{ij}) \in M_{m\times m}$, i.e. we consider the matrix of the first m columns. Then we suppose that we have $(A_m)^t B_m = I_m$, therefore we must have $det(A_m) \neq 0$. Now, since $n > m$, we can consider the homogeneous system with respect to the 'unknowns' $b_{1n}, b_{2n}, ..., b_{mn}$:

$$c_{in} = \sum_{k=1}^{n} a_{ik}b_{kn} = 0 \text{ for } i = 1, 2, ..., m.$$

From which, since $det(A_m) \neq 0$, we obtain that $b_{1n} = b_{2n} = ... = b_{mn} = 0$. Using this fact on the last equation, on the same unknowns, $c_{nn} = \sum_{k=1}^{n} a_{nk}b_{kn} = 1$ we have 0=1, absurd.

□

4. H_v-Lie Algebras and Applications

Recall some definitions from [3].[6],[19]:

Definition 4.1. Let $A = (a_{ij}) \in M_{m\times n}$ be matrix and $s, t \in N$ be naturals such that $1 \leq s \leq m, 1 \leq t \leq n$. Then we define the characteristic-like map $\underline{cst}$ from $M_{m\times n}$ to $M_{s\times t}$ by corresponding to A the matrix $\mathbf{A}\underline{cst} = (a_{ij})$ where $1 \leq i \leq s, 1 \leq j \leq t$. We call this map cut-projection of type $\underline{\text{st}}$. In other words $\mathbf{A}\underline{cst}$ is a matrix obtained from A by cutting the lines, with index greater than s, and columns, with index greater than t.

We can use cut-projections on several types of matrices to define sums and products, however, in this case we have ordinary operations, not multivalued.

In the same attitude we define hopes on any type of matrices:

Definition 4.2. Let $A = (a_{ij}) \in M_{m\times n}$ be matrix and $s, t \in N$, such that $1 \leq s \leq m, 1 \leq t \leq n$. We define the mod-like map $\underline{\text{st}}$ from $M_{m\times n}$ to $M_{s\times t}$ by corresponding to A the matrix $\mathbf{A}\underline{st} = (a_{ij})$ which has as entries the sets

$$a_{ij} = \{a_{i+\kappa s, j+\lambda t} | 1 \leq i \leq s, 1 \leq j \leq t, \text{ and } \kappa, \lambda \in N, i + \kappa s \leq m, j + \lambda t \leq n\}.$$

We call this multivalued map *helix-projection* of type $\underline{\text{st}}$. Thus $\mathbf{A}\underline{st}$ is a set of $s \times t$-matrices $X = (x_{ij})$ such that $x_{ij} \in \underline{a}_{ij}, \forall i, j$. Obviously $\mathbf{A}\underline{mn} = \mathbf{A}$. We may define helix-projections on 'matrices' of which their entries are sets.

Let $\mathbf{A} = (a_{ij}) \in \mathbf{M}_{m\times n}$ be matrix and $s, t \in \mathbf{N}$, such that $1 \leq s \leq m, 1 \leq t \leq n$. Then it is clear that

$$(\mathbf{A}\underline{sn})\underline{st} = (\mathbf{A}\underline{mt})\underline{st} = \mathbf{A}\underline{st}.$$

Let $\mathbf{A} = (a_{ij}) \in \mathbf{M}_{m\times n}$ be matrix and $s, t \in \mathbf{N}$, such that $1 \leq s \leq m, 1 \leq t \leq n$. Then if $\mathbf{A}\underline{st}$ is not a set of matrices but one single matrix then we call $\mathbf{A}$ *cut-helix matrix* of type $s \times t$. In other words $\mathbf{A}$ is a helix matrix of type $s \times t$, if $\mathbf{A}\underline{cst} = \mathbf{A}\underline{st}$.

Definition 4.3. Let $\mathbf{A} = (a_{ij}) \in \mathbf{M}_{m\times n}$ and $\mathbf{B} = (b_{ij}) \in \mathbf{M}_{u\times v}$ be matrices and $s = min(m, u)$, $t = min(n, u)$. We define a hope, called *helix-addition* or *helix-sum*, as follows:

$$\oplus : \mathbf{M}_{m\times n} \times \mathbf{M}_{u\times v} \to \mathbf{P}(\mathbf{M}_{s\times t}) :$$
$$(\mathbf{A}, \mathbf{B}) \to \mathbf{A} \oplus \mathbf{B} = \mathbf{A}\underline{st} + \mathbf{B}\underline{st} = (\underline{\mathbf{a}}_{ij}) + (\underline{\mathbf{b}}_{ij}) \subset \mathbf{M}_{s\times t},$$

where

$$(\underline{\mathbf{a}}_{ij}) + (\underline{\mathbf{b}}_{ij}) = \{(c_{ij}) = (a_{ij} + b_{ij}) | a_{ij} \in \underline{\mathbf{a}}_{ij} \text{ and } b_{ij} \in \underline{\mathbf{b}}_{ij}\}.$$

Let $\mathbf{A} = (a_{ij}) \in \mathbf{M}_{m\times n}$ and $\mathbf{B} = (b_{ij}) \in \mathbf{M}_{u\times v}$ be two matrices and $s = min(m, u)$. We define a hope, called *helix-multiplication or helix-product*, as follows:

$$\otimes : \mathbf{M}_{m\times n} \times \mathbf{M}_{u\times v} \to \mathbf{P}(\mathbf{M}_{m\times v}) :$$
$$(\mathbf{A}, \mathbf{B}) \to \mathbf{A} \otimes \mathbf{B} = \mathbf{A}\underline{ms} \cdot \mathbf{B}\underline{sv} = (\underline{\mathbf{a}}_{ij}) \cdot (\underline{\mathbf{b}}_{ij}) \subset \mathbf{M}_{m\times v},$$

where

$$(\underline{\mathbf{a}}_{ij}) \cdot (\underline{\mathbf{b}}_{ij}) = \left\{(c_{ij}) = (\sum a_{it} b_{tj}) | a_{ij} \in \underline{\mathbf{a}}_{ij} \text{ and } b_{ij} \in \underline{\mathbf{b}}_{ij}\right\}.$$

For the helix-multiplication we remark that we have $\mathbf{A} \otimes \mathbf{B} = \mathbf{A}\underline{ms} \cdot \mathbf{B}\underline{sv}$ so we have either $\mathbf{A}\underline{ms} = \mathbf{A}$ or $\mathbf{B}\underline{sv} = \mathbf{B}$, that means that the helix-projection was applied only in one matrix and only in the rows or in the columns.

The commutativity is valid in the helix-addition. If the appropriate matrices in the helix-sum and in the helix-product are cut-helix, then the result is singleton.

Remark. From the fact that the helix-product on non square matrices is defined, the definition of a Lie-bracket is immediate, therefore the *helix-Lie Algebra* is defined, as well. This algebra is an H_v-Lie Algebra where the fundamental relation ϵ^* gives, by a quotient, a Lie algebra, from which a classification is obtained.

In the following we restrict ourselves on the matrices $\mathbf{M}_{m\times n}$ where $m < n$. Obviously we have analogous results in the case where $m > n$ and for $m = n$ we have the classical theory.

In order to simplify the notation, since we have results on modm, we use:

Notation. For given $\kappa \in \mathbb{N} - \{0\}$, we denote by $\underline{\kappa}$ the remainder resulting from its division by m if the remainder is non zero, and $\underline{\kappa} = m$ if the remainder is zero. Thus a matrix $\mathbf{A} = (a_{\kappa\lambda}) \in \mathbf{M}_{m\times n}$, $m < n$ is a cut-helix if we have $a_{\kappa\lambda} = a_{\kappa\underline{\lambda}} \forall \kappa, \lambda \in \mathbb{N} - \{0\}$. Moreover let us denote by $\mathbf{I}_c = (c_{\kappa\lambda})$ the *cut-helix unit matrix* which the cut matrix is the unit matrix $\mathbf{I}_m$. Therefore, since $\mathbf{I}_m$=$(\delta_{\kappa\lambda})$, where $\delta_{\kappa\lambda}$ is the Kroneckers delta, we obtain that, $\forall \kappa, \lambda$, we have $c_{\kappa\lambda} = \delta_{\kappa\underline{\lambda}}$.

Proposition 4.4. *For $m < n$ in $(M_{m\times n}, \otimes)$ the cut-helix unit matrix $\mathbf{I}_c = (c_{\kappa\lambda})$, where $c_{\kappa\lambda} = \delta_{\kappa\lambda}$, is a left scalar unit and a right unit. It is the only one left scalar unit.*

Proof. Let $\mathbf{A}, \mathbf{B} \in M_{m\times n}$ then in the helix-multiplication, since $m < n$, we take helix projection of $\mathbf{A}$, therefore the result $\mathbf{A}\otimes\mathbf{B}$ is singleton if $\mathbf{A}$ is a cut-helix matrix of type $m \times m$. Moreover in order to have $\mathbf{A} \otimes \mathbf{B} = \mathbf{A}_{\underline{mm}} \cdot \mathbf{B} = \mathbf{B}$, the $\mathbf{A}_{\underline{mm}}$ must be the unit matrix. Consequently $\mathbf{I}_c = (c_{\kappa\lambda})$, where $c_{\kappa\lambda} = \delta_{\kappa\underline{\lambda}}$, $\forall \kappa, \lambda \in \mathbb{N} - \{0\}$, is necessarily the left scalar unit element.

Now we remark that it is not possible to have the same case for the right matrix $\mathbf{B}$, therefore we have only to prove that $\mathbf{I}_c$ is a right unit but it is not a scalar, consequently it is not unique.

Let $\mathbf{A} = (a_{uv}) \in \mathbf{M}_{m\times n}$ and consider the hyperproduct $\mathbf{A} \otimes \mathbf{I}_c$. In the entry $\kappa\lambda$ of this hyperproduct there are sets, for all $1 \leq \kappa \leq m$, $1 \leq \lambda \leq n$, of the form

$$\sum \underline{a}_{\kappa s} c_{s\lambda} = \sum \underline{a}_{\kappa s} \delta_{s\lambda} = \underline{a}_{\kappa\underline{\lambda}} \ni a_{\kappa\lambda}.$$

Therefore $\mathbf{A} \otimes \mathbf{I}_c \ni \mathbf{A}, \forall \mathbf{A} \in \mathbf{M}_{m\times n}$. □

In the following examples we denote $\mathbf{E}_{ij}$ any type of matrices which have the ij-entry 1 and in all the other entries we have 0.

Construction 4.5. [6] Consider the 2×3 matrices of the following form, for $\kappa \in \mathbb{N}$,

$$\mathbf{A}_\kappa = \mathbf{E}_{11} + \kappa\mathbf{E}_{21} + \mathbf{E}_{22} + \mathbf{E}_{23},\ \mathbf{B}_\kappa = \kappa\mathbf{E}_{21} + \mathbf{E}_{22} + \mathbf{E}_{23}$$

Then we obtain $\mathbf{A}_\kappa \otimes \mathbf{A}_\lambda = \{\mathbf{A}_{\kappa+\lambda}, \mathbf{A}_{\lambda+1}, \mathbf{B}_{\kappa+\lambda}, \mathbf{B}_{\lambda+1}\}$.
Similarly, we have $\mathbf{B}_\kappa \otimes \mathbf{A}_\lambda = \{\mathbf{B}_{\kappa+\lambda}, \mathbf{B}_{\lambda+1}\}$,$\mathbf{A}_\kappa \otimes \mathbf{B}_\lambda = \mathbf{B}_\lambda = \mathbf{B}_\kappa \otimes \mathbf{B}_\lambda$.

Thus the set $\{\mathbf{A}_\kappa, \mathbf{B}_\lambda | \kappa, \lambda \in \mathbb{N}\}$ becomes an H_v-semigroup which is not COW because for $\kappa \neq \lambda$ we have $\mathbf{B}_\kappa \otimes \mathbf{B}_\lambda = \mathbf{B}_\lambda \neq \mathbf{B}_\kappa = \mathbf{B}_\lambda \otimes \mathbf{B}$, however

$$(\mathbf{A}_\kappa \otimes \mathbf{A}_\lambda) \cap (\mathbf{A}_\lambda \otimes \mathbf{A}_\kappa) = \{\mathbf{A}_{\kappa+\lambda}, \mathbf{B}_{\kappa+\lambda}\} \neq \varnothing, \forall \kappa, \lambda \in \mathbb{N}.$$

All elements $\mathbf{B}_\lambda$ are right absorbing and $\mathbf{B}_1$ is a left scalar element, because $\mathbf{B}_1 \otimes \mathbf{A}_\lambda = \mathbf{B}_{\lambda+1}$ and $\mathbf{B}_1 \otimes \mathbf{B}_\lambda = \mathbf{B}_\lambda$. The element $\mathbf{A}_0$ is a unit.

Construction 4.6. Consider the 2×3 matrices of the following form, for $\kappa \in \mathbb{N}$,

$$\mathbf{A}_{\kappa\lambda} = \mathbf{E}_{11} + \mathbf{E}_{13} + \kappa\mathbf{E}_{21} + \mathbf{E}_{22} + \lambda\mathbf{E}_{23}.$$

Then we obtain $\mathbf{A}_{\kappa\lambda} \otimes \mathbf{A}_{st} = \{\mathbf{A}_{\kappa+s,\kappa+t}, \mathbf{A}_{\kappa+s,\lambda+t}, \mathbf{A}_{\lambda+s,\kappa+t}, \mathbf{A}_{\lambda+s,\lambda+t}\}$.
Moreover $\mathbf{A}_{st} \otimes \mathbf{A}_{\kappa\lambda} = \{\mathbf{A}_{\kappa+s,\lambda+s}, \mathbf{A}_{\kappa+s,\lambda+t}, \mathbf{A}_{\kappa+t,\lambda+s}, \mathbf{A}_{\kappa+t,\lambda+t}\}$, so
$\mathbf{A}_{\kappa\lambda} \otimes \mathbf{A}_{st} \cap \mathbf{A}_{st} \otimes \mathbf{A}_{\kappa\lambda} = \{\mathbf{A}_{\kappa+s,\lambda+t}\}$, thus $(\otimes)$ is COW. The helix multiplication $(\otimes)$ is associative.

Consider all $m \times n$ matrices with $m \leq n$ and we write these matrices as block matrices of the form $\mathbf{A} = (\underline{\mathbf{A}}|\underline{\mathbf{A}'})$, where $\underline{\mathbf{A}}$ be a square $m \times m$ matrix and $\underline{\mathbf{A}'}$ be of type $m \times (n-m)$. Denote $M_{\underline{m}\times\underline{n}}$ the set of all $m \times n$ matrices (with $m \leq n$) such that in every A the square matrix $\underline{\mathbf{A}}$ is invertible. Take any $\mathbf{P} \subset \mathbf{M}_{\underline{m}\times\underline{n}}$ and define a P-hope as follows:

$$\mathbf{A}^\circ\mathbf{B} = \mathbf{A}\mathbf{P}^t\mathbf{B}, \forall \mathbf{A}, \mathbf{B} \in \mathbf{M}_{\underline{m}\times\underline{n}}$$

where $\mathbf{P}^t$ is the set of all transpose matrices from the set $\mathbf{P}$. Then the $(\mathbf{M}_{\underline{m}\times\underline{n}}, +, \circ)$ becomes a multiplicative hyperring where all matrices of type $\mathbf{A}_e = (\underline{\mathbf{A}}^{-1}|\underline{0})$, for $\mathbf{A} \in \mathbf{P}$ are left units. Indeed

$$\mathbf{A}_e^\circ\mathbf{B} = \mathbf{A}_e P^t\mathbf{B} \ni \mathbf{A}_e\mathbf{A}\mathbf{B} = I_{m\times m}\mathbf{B} = \mathbf{B}, \forall \mathbf{B} \in \mathbf{M}_{\underline{m}\times\underline{n}}$$

During last decades the hyperstructures have a variety of applications in other branches of mathematics and in many other sciences. These applications range from biomathematics -conchology, inheritance- and hadronic physics to mention but a few. The hyperstructures theory is closely related to fuzzy theory; consequently, hyperstructures can now be widely applicable in industry and production, too. In several books and papers [1], [2], [5], [7], [18], [20], one can find numerous applications.

The Lie-Santilli theory on *isotopies* was born in 1970's to solve Hadronic Mechanics problems. Santilli proposed a 'lifting' of the n-dimensional trivial unit matrix of a normal theory into a nowhere singular, symmetric, real-valued, positive-defined, n-dimensional new matrix. The original theory is reconstructed such as to admit the new matrix as left and right unit. The isofields needed in this theory correspond into the hyperstructures were introduced by Santilli and Vougiouklis in 1996 and they are called e-hyperfields [2], [5]. The H_v-fields can give e-hyperfields which can be used in the isotopy theory for applications in several fields as in physics or biology. We present in the following the main definitions and results restricted in the H_v-structures.

Definition 4.7. A hyperstructure $(\mathbf{H}, \cdot)$ which contain a unique scalar unit e, is called *e-hyperstructure*, where we assume that $\forall x$, there exists an inverse x^{-1}, i.e. $e \in x \cdot x^{-1} \cap x^{-1} \cdot x$. The inverses are not necessarily unique.A hyperstructure $(F, +, \cdot)$, where $(+)$ is an operation and $(\cdot)$ is a hope, is called *e-hyperfield* if the following axioms are valid:

(a) $(F, +)$ is an abelian group with the additive unit 0,

(b) $(\cdot)$ is WASS,
(c) $(\cdot)$ is weak distributive with respect to $(+)$,
(d) 0 is absorbing element: $0 \cdot x = x \cdot 0 = 0, \forall x \in F$,
(e) there exist a multiplicative scalar unit 1, i.e. $1 \cdot x = x \cdot 1 = x, \forall x \in F$, and
(f) for all $x \in F$ there exists a unique inverse x^{-1}, such that $1 \in x \cdot x^{-1} \cap x^{-1} \cdot x$.

The elements of an e-hyperfield are called *e-hypernumbers*. In the case that the relation: $1 = x \cdot x^{-1} = x^{-1} \cdot x$, is valid, then we say that we have a *strong e-hyperfield*.

A general construction based on the partial ordering of the H_v-structures:

Theorem 4.8. *The Main e-Construction*[5],[6]. *Given a group $(G, \cdot)$, where e is the unit, then we define in G, a large number of hopes $(\otimes)$ by extending $(\cdot)$ as follows:*

$$x \otimes y = \{xy, g_1, g_2, ...\}, \forall x, y \in G - \{e\}, \text{ and } g_1, g_2, ... \in G - \{e\}$$

$g_1, g_2,...$ *are not necessarily the same for each pair (x,y). Then $(G, \otimes)$ becomes an H_v-group, actually is an H_b-group which contains the $(G, \cdot)$. The H_v-group $(G, \otimes)$ is an e-hypergroup. Moreover, if for each x,y such that $xy = e$, so we have $x \otimes y = xy$, then $(G, \otimes)$ becomes a strong e-hypergroup*

The proof is immediate since we enlarge the results of the group by putting elements from G and applying the Little Theorem. Moreover one can see that the unit e is a unique scalar and for each x in G, there exists a unique inverse x^{-1}, such that $1 \in x \cdot x^{-1} \cap x^{-1} \cdot x$ and if this condition is valid then we have $1 = x \cdot x^{-1} = x^{-1} \cdot x$. So the hyperstructure $(G, \otimes)$ is a strong e-hypergroup.

An application combining hyperstructures and fuzzy theory, is to replace the scale of Likert in questionnaires by the bar of Vougiouklis & Vougiouklis [20]:

Definition 4.9. In every question substitute the Likert scale with 'the bar' whose poles are defined with '0' on the left end, and '1' on the right end:

0 ____________________ 1

The subjects/participants are asked instead of deciding and checking a specific grade on the scale, to cut the bar at any point s/he feels expresses her/his answer to the specific question.

The use of the bar of Vougiouklis & Vougiouklis instead of a scale of Likert has several advantages during both the filling-in and the research processing [20]. The final suggested length of the bar, according to the Golden Ratio, is 6.2cm.

As a conclusion, we remark that: since the class of H_v-structures is extremely large, they can be used in other sciences as organized devices.

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