# REDUCIBILITY IN HYPERGROUPS (CRISP AND FUZZY CASE)

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Abstract. In this paper we propose an extension of the concept of reducibility in a crisp hypergroup to the fuzzy case. The idea will be developed in two directions: defining the reducibility in a fuzzy hypergroup and defining the fuzzy reducibility in a crisp hypergroup endowed with a fuzzy set.

## 1. Introduction

It may happen that a hyperproduct on a given set $H$ does not discriminate between a pair of elements of $H$, when the elements play interchangeable roles with respect to the hyperoperation. Thus a certain equivalence relation can be defined in order to identify the elements with the same properties. This characteristic of hypergroups has been noticed by Jantosciak [12] in 1990, who described it by the meaning of three equivalences, called *fundamental relations*. Starting from the classical (or crisp) case, here we will extend these notions to the fuzzy case in two different directions: defining similar relations in a fuzzy hypergroup and in a crisp hypergroup endowed with a fuzzy set.

Then a *reduced hypergroup* is a hypergroup having the equivalence class of each element with respect to the essentially indistinguishable relation a singleton. Therefore, the study of the hypergroups may be divided in the study of the reduced hypergroups and in that of the hypergroups with the same reduced form, as Jantosciak proved in [12]. Necessary and sufficient conditions such that a hypergroup associated with a binary relation (in the sense of Rosenberg [13]) is a reduced hypergroup have been presented in the papers [7, 8, 9]. Moreover, in [8] some aspects concerning the direct product of the reduced hypergroups, some properties of the hypergroups associated with the intersection, union, or composition of two binary relations defined on the same set have been investigated. The same properties of the fundamental equivalences defined by Jantosciak can be described in terms of

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Boolean matrices [10, 11], thus allowing to create new algorithms to compute the number of the hypergroups associated with binary relations.

This idea will be also developed in the fuzzy case in two different directions: studying the indistinguishability between the elements of a fuzzy hypergroup, or between the images of the elements of a crisp hypergroup endowed with a fuzzy set. In particular, we will introduce the notions of *reduced fuzzy hypergroup*, i.e. a fuzzy hypergroup which is reduced, and that of *fuzzy reduced hypergroup*, that is a crisp hypergroup endowed with a fuzzy set which is fuzzy reduced. In order to do these, new equivalence relations (similarly to those of operationally equivalence, inseparability and essentially indistinguishability defined by Jantosciak [12]) will be defined and discussed.

## 2. **Preliminaries**

This section is dedicated to a short review of the basic notions and results used in the following. In particular we will deal with regular and strongly regular relations, i.p.s. and complete hypergroups, fuzzy hypergroups and crisp hypergroups associated with fuzzy sets.

We start we (strongly) regular relations, so those equivalences that connect algebraic hyperstructures with corrisponding algebraic structures defined on the quotient support set of the hyperstructure.

**Definition 2.1.** An equivalence relation $R$ on a hypergroupoid $(H, \circ)$ is called *regular to the right* if, for any $(x, y) \in H^2$, we have the following implication:
$xRy \Longrightarrow \forall u \in x \circ a, \exists v \in y \circ a$ such that $uRv$ and $\forall v' \in y \circ a, \exists u' \in x \circ a$ such that $u'Rv'$, for any $a \in H$.

We say that $R$ is *strongly regular to the right* if, for any $(x, y) \in H^2$, we have the implication: $xRy \Longrightarrow \forall u \in x \circ a, \forall v \in y \circ a$ we have $uRv$, for any $a \in H$. Similarly, we define the *(strong) regularity to the left.*

An equivalence relation $R$ is *(strongly) regular* if it is (strongly) regular to the left and to the right.

As it was said before, these relations are useful to obtain algebraic structures from hyperstructures. More exactly, starting with a (semi)hypergroup $H$ and using a strongly regular relation $R$ on $H$, then the quotient $H/R$ is a (semi)group.

Since we will work in the next sections with i.p.s. hypergroups and complete hypergroups, we recall here their definition and main properties.

**Definition 2.2.** A hypergroup $(H, \circ)$ is called *canonical*, if it satisfies the following conditions:

1) it is commutative;
2) it has a scalar identity 0 such that $0 \circ x = x$, for any $x \in H$;
3) every element $x \in H$ has a unique inverse $x^{-1} \in H$, that is $0 \in x \circ x^{-1}$;
4) it is reversible, so $y \in a \circ x \Longrightarrow x \in a^{-1} \circ y$, for any $a, x, y \in H$.

An *i.p.s.hypergroup (hypergroup with partial scalar identities)* is a canonical hypergroup verifying the suplementary property: $x \in x \circ a \Longrightarrow a \circ x = x$, for all $a, x \in H$.

In the next result we summarize the main properties of i.p.s. hypergroups, useful in our research.

**Proposition 2.3.** ([1, 2, 3]) *Let $(H,\circ)$ be an i.p.s. hypergroup.*

1) *For any $x \in H$, the set $x \circ x^{-1}$ is a subgroup of $H$.*
2) *For any $x \in H \setminus \{0\}$, we have: or $x \in Sc(H)$ (i.e. $|x \circ y| = 1$, for all $y \in H$, meaning that $x$ is a scalar of $H$) or there exists $u \in Sc(H) \setminus \{0\}$ such that $u \in x \circ x^{-1}$. Moreover $|Sc(H)| \geq 2$.*
3) *If $x \in Sc(H)$, then the set of partial scalar identities of $x$, denoted by $I_{ps}(x)$ contains just $0$.*
   *If $x \notin Sc(H)$, then $I_{ps}(x) \subset Sc(H) \cap x \circ x^{-1}$ and therefore $|I_{ps}(x)| \geq 2$.*
4) *For any $u \in Sc(H)$ and any $x \in H$, there exists a unique $y \in H$ such that $u \in x \circ y$. It follows that $Sc(H) \subset I_{ps}(x)$, for any $x \in H \setminus Sc(H)$.*

In practice, instead of the definition of a complete hypergroup, the following characterization is used.

**Theorem 2.4.** *A hypergroup $(H,\circ)$ is complete if it can be written as the union of its subsets $H = \bigcup_{g \in G} A_g$, where*

1) *$(G,\cdot)$ is a group.*
2) *For any $(g_1,g_2) \in G^2$, $g_1 \neq g_2$, we have $A_{g_1} \cap A_{g_2} = \emptyset$.*
3) *If $(a,b) \in A_{g_1} \times A_{g_2}$, then $a \circ b = A_{g_1 g_2}$.*

**Proposition 2.5.**

1) *Any complete hypergroup is a regular reversible hypergroup.*
2) *Any complete commutative hypergroup is a join space.*
3) *If $H$ is a complete hypergroup, then the heart $\omega_H$ is the set of all identities of $H$.*
4) *Let $H$ be a complete hypergroup and $H = \bigcup_{g \in G} A_g$, with $(G,\cdot)$ a group with the identity element $e$. Then $A_e = \omega_H$.*

Denote by $\mathcal{F}(H)$ the set of all fuzzy subsets of a nonempty set $H$.

**Definition 2.6.** (M.K. Sen et al., 2007) A fuzzy hyperoperation on $H$ is a mapping $\circ : H \times H \to \mathcal{F}(H)$ and the pair $(H,\circ)$ is called a *fuzzy hypergroupoid.* $(H,\circ)$ is called a *fuzzy semihypergroup* if, $\forall a,b,c \in H, (a \circ b) \circ c = a \circ (b \circ c)$, where for any fuzzy subset $\mu$ of $H$ and $\forall r \in H$,

$$(a \circ \mu)(r) = \begin{cases} \bigvee_{t \in H}((a \circ t)(r) \wedge \mu(t), & \text{if } \ \mu \neq 0 \\ 0, & \text{otherwise} \end{cases}$$

and

$$(\mu \circ a)(r) = \begin{cases} \bigvee_{t \in H}(\mu(t) \wedge (t \circ a)(r)), & \text{if } \ \mu \neq 0 \\ 0, & \text{otherwise} \end{cases}$$

A fuzzy semihypergroup is called a *fuzzy hypergroup* if $a \circ H = H \circ a = \chi_H$ (the characteristic function of $H$), for any $a \in H$.

We conclude this section with the connections between crisp hypergroups and fuzzy sets, established by Corsini in[4, 6]. First, to any crisp hypergroupoid $(H, \circ)$ we may associate a fuzzy set $\widetilde{\mu}$ as it follows: for any $u \in H$, one considers

$$\widetilde{\mu}(u) = \frac{\displaystyle\sum_{(x,y)\in Q(u)} \frac{1}{|x \circ y|}}{q(u)}, \tag{1}$$

where $Q(u) = \{(a, b) \in H^2 \mid u \in a \circ b\}, \quad q(u) = |Q(u)|$. If $Q(u) = \emptyset$, set $\widetilde{\mu}(u) = 0$.

In other words, $\widetilde{\mu}(u)$ is the average value of the reciprocals of the sizes of $x \circ y$, for all $x \circ y$ containing $u$.

On the other hand, with any hypergroupoid $H$ endowed with a fuzzy set $\alpha$, we can associate a join space $(H, \circ_\alpha)$ as follows (see [4]): for any $(x, y) \in H^2$,

$$x \circ_\alpha y = \{z \in H \mid \alpha(x) \wedge \alpha(y) \leq \alpha(z) \leq \alpha(x) \vee \alpha(y)\}. \tag{2}$$

Combining both constructions, we may obtain by iteration a sequence of fuzzy sets and join spaces having the length called *fuzzy grade*. For this reason the fuzzy set $\widetilde{\mu}$ is called the *grade fuzzy set* associated with a non-empty set $H$.

## 3. Fundamental Relations and Reduced Hypergroups

In hyperstructure theory some equivalences play a fundamental role in obtaining quotient structures, being called *fundamental relations*. They can be devided into two separate groups: the first one contains the relations $\alpha, \beta, \gamma$ defined on a hyperring (the first one) and on a semihypergroup (the other two) such that the quotient structure is a ring, a semigroup and respectively a commutative semigroup. In the second group we include the fundamental relations defined by Jantosciak in order to define the notion of reduced hypergroup.

**Definition 3.1.** (J. Jantosciak, 1991) Two elements $x, y$ in a hypergroup $(H, \circ)$ are called:

(1) *operationally equivalent* or by short *o-equivalent*, and write $x \sim_o y$, if $x \circ a = y \circ a$, and $a \circ x = a \circ y$, for any $a \in H$.
(2) *inseparable* or by short *i-equivalent*, and write $x \sim_i y$, if, for all $a, b \in H$, $x \in a \circ b \Longleftrightarrow y \in a \circ b$.
(3) *essentially indistinguishable* or by short *e-equivalent* if they are operationally equivalent and inseparable.

**Example 3.2.** Let $H = \{a, b, c, d\}$. Let the hyperoperation $\circ$ on $H$ be given by the following table

| $H$ | $a$ | $b$ | $c$ | $d$ |
|---|---|---|---|---|
| $a$ | $a, b$ | $a, b$ | $c, d$ | $c, d$ |
| $b$ | $a, b$ | $a, b$ | $c, d$ | $c, d$ |
| $c$ | $c, d$ | $c, d$ | $a$ | $b$ |
| $d$ | $c, d$ | $c, d$ | $b$ | $a$ |

One easily checks that $(H, \circ)$ is a hypergroup and

- the o-equivalence classes are: $\hat{a}_o = \hat{b}_o = \{a, b\}$, $\hat{c}_o = \{c\}$, $\hat{d}_o = \{d\}$;
- the i-equivalence classes are: $\hat{a}_i = \{a\}$, $\hat{b}_i = \{b\}$, $\hat{c}_i = \hat{d}_i = \{c, d\}$;
- the e-equivalence classes are: $\hat{a}_e = \{a\}$, $\hat{b}_e = \{b\}$, $\hat{c}_e = \{c\}$, $\hat{d}_e = \{d\}$.

**Definition 3.3.** (J. Jantosciak, 1991) A *reduced hypergroup* has the equivalence class of each element with respect to the essentially indistinguishable relation $\sim_e$ a singleton.

Using the fundamental relations on a hypergroup introduced by Jantosciak [12], the study of the hypergroups can be divided into two parts: the study of reduced hypergroups and the study of all hypergroups having the same reduced form $H/{\sim_e}$.

**Example 3.4.** Let $H = \mathbb{Z} \times \mathbb{Z}^*$. Let $c$ be the equivalence relation that puts equivalent fractions into classes:

$$\widehat{(x, y)}_c = \{(u, v) \mid xv = yu\}, \forall (x, y) \in H^2.$$

Defining on $H$ the hyperoperation

$$(w, x) \circ (y, z) = \widehat{(wz + xy, xz)}_c$$

we get that $(H, \circ)$ is a hypergroup in which

$$\widehat{(x, y)}_o = \widehat{(x, y)}_i = \widehat{(x, y)}_e = \widehat{(x, y)}_c,$$

so $H$ is not a reduced hypergroup, while the reduced form $H/\sim_e$ is isomorphic with the set of rationals $\mathbb{Q}$.

**Theorem 3.5.** (Jantosciak, 1991) *For any hypergroup $H$, the quotient hypergroup $H/{\sim_e}$ is a reduced hypergroup, called the reduced form of $H$.*

To each hypergroup $H$ there corresponds canonically a reduced hypergroup $H/{\sim_e}$ in such a way that the hyperoperation on $H$ may be reconstructed from the hyperoperation on $H/{\sim_e}$: if $f$ is the canonical homomorphism from $H$ to $H/{\sim_e}$, then $x \circ y = f^{-1}(f(x)f(y))$.

Given a reduced hypergroup $H$, we wish to determine all hypergroups having $H$ as their reduced form.

**Theorem 3.6.** (Jantosciak, 1991) *Let $H$ be a hypergroup, $K$ be a set and $f$ a mapping of $K$ onto $H$. Defining on $K$ the hyperoperation $x \cdot y = f^{-1}(f(x)f(y))$, we obtain that $K$ is a hypergroup (known as $K_H$-hypergroup) and $H$ is reduced if and only if $K/\sim_e \simeq H$.*

Regarding the regularity of the fundamental relations, we know the following properties.

**Theorem 3.7.** (Cristea et al. [11])

(1) *The operational equivalence is regular, but in general not strongly regular.*
(2) *The inseparability is not a regular relation.*
(3) *The essential indistinguishability is regular, but in general not strongly regular.*

**Theorem 3.8.** (Cristea et al. [11]) *The inseparability relation* $\sim_i$ *is regular on a hypergroupoid* $(H, \circ)$ *if and only if* $a \sim_i b \Longrightarrow a \sim_o b$*, for* $a, b \in H$*.*

We conclude this section with the study of reducibility of two classes of hypergroups: the complete hypergroups and the i.p.s. hypergroups.

**Theorem 3.9.** *Any i.p.s. hypergroup is reduced.*

*Proof.* If $H$ is a hypergroup with a scalar identity, it is clear that we have the following equivalences:

$$a \sim_o b \Longleftrightarrow a \sim_i b \Longleftrightarrow a \sim_e b \Longleftrightarrow a = b$$

and therefore the hypergroup is reduced. □

**Theorem 3.10.** *Any proper complete hypergroup is not reduced.*

*Proof.* Let $(H, \circ)$ be a complete hypergroup, where $H = \bigcup_{g \in G} A_g$, with $(G, \cdot)$ a group.We define the equivalence $\sim$ on $H$ by taking

$$x \sim y \Longleftrightarrow \exists g \in G : x, y \in A_g.$$

First we prove that the relations $\sim$ and $\sim_e$ (the essentially indistinguishability relation) are the same.

By the definition of a complete hypergroup, for any $x \in G$, there exists a unique $g_x \in$ such that $x \in A_{g_x}$ and then it is clear that $\sim = \sim_e$:

"$\subseteq$": Consider $x \sim y$, i.e. there exists a unique $g_x = g_y \in G$ such that $x, y \in A_{g_x}$. For any $a \in H$ (in particular $a \in A_{g_a}$) we have $x \circ a = A_{g_x g_a} = y \circ a$, thus $x \sim_o y$. On the other hand, if $x \in a \circ b = A_{g_a g_b} \cap A_{g_x}$, then $g_a g_b = g_x = g_y$ and therefore $y \in a \circ b$ and vice versa. Threby $x \sim_i y$. Combining the two results, it follows that $x \sim_e y$.

"$\supseteq$": Conversely, suppose that $x \sim_e y$, so $x \sim_o y$ and $x \sim_i y$, meaning that $x \in a \circ b$ if and only if $y \in a \circ b$. It follows that $x, y \in A_{g_a g_b}$ and therefore there exists $g_x = g_a g_b \in G$ such that $x, y \in A_{g_x}$, so $x \sim y$.

Now we can conclude the proof. Since $(H, \circ)$ is proper complete hypergroup, there exists at least one $g \in G$ such that $|A_g| \geq 2$, i.e. there exist $a \neq b \in H$ such that $a \sim b$. Then $asim_e b$, meaning that $H$ is not reduced. □

## 4. Reducibility of the Hypergroups Associated with Binary Relations

Till now, various hyperoperations have been defined using a binary relation $\rho$ on a nonempty set $H$. We recall here those introduced by Rosenberg [13] in 1998, Corsini [5] in 2000 and Cristea et al. [11] in 2011.

**Definition 4.1.** Let $\rho$ be a binary relation on a nonempty set $H$. For any $x \in H$, set:

$$L(x) = \{y \in H \mid (y, x) \in \rho\}, R(x) = \{z \in H \mid (x, z) \in \rho\}.$$

The following types of hyperproducts can be defined:

- Rosenberg's hyperoperation [13]: $x \circ y = R(x) \cup R(y)$

- Corsini's hyperoperation [5]: $x \odot y = R(x) \cap L(y)$
- Cristea's hyperoperation [11]: $x \otimes y = L(x) \cup R(y)$

Necessary and sufficient conditions such that the obtained hypergroupoids are reduced hypergroups are given in the following results given by Cristea et al.

**Theorem 4.2.** [7] *The hypergroup $\mathbb{H}_\rho = (H, \circ)$ obtained by Rosenberg's hyperoperation is reduced if and only if, for any $x, y \in H$, $x$ different from $y$, either $L_x \neq L_y$ or $R_x \neq R_y$.*

**Theorem 4.3.** [5, 11] *The hyperstructure $\mathcal{H}_\rho = (H, \otimes_\rho)$ endowed with Corsini's hyperoperation is a hypergroupoid if and only if $\rho^2 = H \times H$. If the hyperoperation $\otimes_\rho$ is left or right reproductive, then $\mathcal{H}_\rho$ is the total hypergroup. So, the unique hypergroup obtained in this manner is the total hypergroup, which clearly is not reduced.*

**Theorem 4.4.** [11] *Let $\rho$ be a binary relation on $H$ with full domain and full range such that, for any $x \in H$, $x \notin L(x)$. Then the $H_v$-group $(H, \otimes_\rho)$, obtained using Cristea's hyperoperation, is reduced if and only if there are no any $x \neq y \in H$ such that $L(x) = L(y)$ and $R(x) = R(y)$.*

## 5. **The Fuzzy Case of Reducibility**

In this section we propose a method of fuzzification of the concept of reducibility in a hypergroup. It is natural to consider it on a fuzzy hypergroup or on a crisp hypergroup endowed with a fuzzy set. In the first case, using fuzzy hyperoperations, we obtain the so called *reduced fuzzy hypergroups*, so fuzzy hypergroups that are reduced. In the second one we define the *fuzzy reduced hypergroups*, i.e. crisp hypergroups that are fuzzy reduced with respect to the associated fuzzy set.

### 5.1. **Reduced Fuzzy Hypergroups.**

**Definition 5.1.** In a fuzzy hypergroup $(H, \circ)$ define the fundamental relations as it follows:

i) $x$ and $y$ are *operationally equivalent* and write $x \sim_o y$, if $x \circ a = y \circ a$ and $a \circ x = a \circ y$, for any $a \in H$, i.e. $(x \circ a)(r) = (y \circ a)(r)$ and $(a \circ x)(r) = (a \circ y)(r)$, for all $a, r \in H$;

ii) $x$ and $y$ are *inseparable* and write $x \sim_i y$, if $x \in supp(a \circ b) \iff y \in supp(a \circ b)$, for $a, b \in H$, that is $(a \circ b)(x) \neq 0 \iff (a \circ b)(y) \neq 0$, for $a, b \in H$;

iii) $x$ and $y$ are *essentially indistinguishable* and write $x \sim_e y$, if they are operationally equivalent and inseparable.

**Definition 5.2.** A fuzzy hypergroup $(H, \circ)$ is a *reduced fuzzy hypergroup* if and only if the equivalence class of each element in $H$ with respect to the essentially indistinguishable relation is a singleton:

$$\forall x \in H, \hat{x}_e = \{x\}.$$

5.2. **Fuzzy Reduced Hypergroups.**

**Definition 5.3.** In a crisp hypergroup $(H, \circ)$ endowed with a fuzzy set $\mu$, we introduce the following equivalences:

i) $x$ and $y$ are *fuzzy operationally equivalent* and write $x \sim_{fo} y$ if, for any $a \in H$, $\mu(x \circ a) = \mu(y \circ a)$;
ii) $x$ and $y$ are *fuzzy inseparable* and write $x \sim_{fi} y$ if $\mu(x) \in \mu(a \circ b) \Longleftrightarrow \mu(y) \in \mu(a \circ b)$, for $a, b \in H$;
iii) $x$ and $y$ are *fuzzy essentially indistinguishable* and write $x \sim_{fe} y$, if they are fuzzy operationally equivalent and fuzzy inseparable.

**Definition 5.4.** The crisp hypergroup $(H, \circ)$ is a *fuzzy reduced hypergroup* if and only if the equivalence class of each element in $H$ with respect to the fuzzy essentially indistinguishable relation is a singleton:

$$\forall x \in H, \hat{x}_{fe} = \{x\}.$$

The regularity of the new introduced relations is discussed in the following examples.

**Example 5.5.** In any hypergroupoid $H$ endowed with a fuzzy set $\mu$, the following implication holds:

$$a \sim_o b \Longrightarrow a \sim_{fo} b.$$

Therefore, if there exist $a \neq b \in H$ such that $a \sim_o b$, (i.e. there exists at least one element $a$ such that its equivalence class $\hat{a}_o$ is not a singleton), then the relation $\sim_{fo}$ is regular (since the relation $\sim_o$ is regular).

**Example 5.6.** Let's consider the hypergroup $(H, \circ)$ represented by the following table:

| $H$ | $a$ | $b$ | $c$ |
|---|---|---|---|
| $a$ | $a$ | $a, b$ | $a, c$ |
| $b$ | | $b$ | $b, c$ |
| $c$ | | | $c$ |

Defining a fuzzy set $\mu$ on $H$ such that $\mu(a) = \mu(b) \neq \mu(c)$, we get that the relation $\sim_{fo}$ is regular.

First, we notice that $a \not\sim_o b$, but $a \sim_{fo} b$. Moreover, $\hat{a}_{fo} = \{a, b\}, \hat{c}_{fo} = \{c\}$.

Since $a \sim_{fo} b$, we will prove that, for any $u \in a \circ x$, there exists $v \in b \circ x$ such that $u \sim_{fo} v$, for any $x \in H$:

*i)* If $x = a$, then for any $u \in a \circ a = a$, there exists $v \in b \circ a = \{a, b\}$ (and we take $v = a$) such that $u \sim_{fo} v$.
*ii)* If $x = b$, then for any $u \in a \circ b = \{a, b\}$, there exists $v \in b \circ b = b$ such that $u \sim_{fo} v$.
*iii)* If $x = c$, then for any $u \in a \circ c = \{a, c\}$, there exists $v \in b \circ c = \{b, c\}$ (if $u = a$, then we take $v = b$, while if $u = c$, then we take $v = c$) such that $u \sim_{fo} v$.

**Example 5.7.** Let us consider $(H, \circ)$ the hypergroupoid defined by the following table:

| $H$ | $a$ | $b$ | $c$ | $d$ |
|---|---|---|---|---|
| $a$ | $a, c$ | $c, d$ | $a, b, c$ | $d$ |
| $b$ | $c, d$ | $a, c$ | $a, b, c$ | $d$ |
| $c$ | $a, b, c$ | $a, b, c,$ | $c$ | $c, d$ |
| $d$ | $d$ | $d$ | $c, d$ | $d$ |

Define on $H$ a fuzzy set $\mu$ such that $\mu(a) = \mu(d) \neq \mu(b) \neq \mu(c)$. We notice that $a \not\sim_o b$, but $a \sim_{fo} b$. Moreover, $\hat{a}_{fo} = \{a, b\}, \hat{c}_{fo} = \{c\}, \hat{d}_{fo} = \{d\}$.

We will prove that the relation $\sim_{fo}$ is not regular.

Indeed, suppose that the relation $\sim_{fo}$ is regular; then, for any $x \in H$, we should have the implication

$$a \sim_{fo} b \Longrightarrow \forall u \in a \circ x, \exists v \in b \circ x : u \sim_{fo} v.$$

If $x = a$, then the relation reads: for any $u \in a \circ a = \{a, c\}$, there exists $v \in b \circ a = \{c, d\}$ such that $u \sim_{fo} v$. For $u = a$ there is no $v \in b \circ a = \{c, d\}$ such that $u \sim_{fo} v$, so the implication is not true, meaning that the relation $\sim_{fo}$ is not regular.

We can conclude that the relation $\sim_{fo}$ is not regular in general.

## 6. Conclusions and Open Problems

In this paper we have investigated the properties of the three fundamental relations introduced by Jantosciak, searching for significant examples of crisp hypergroups and fuzzy hypergroups that are fuzzy reduced or reduced, respectively. For example, we have proven that a hypergroup with partial scalar identities (i.e. an i.p.s. hypergroup) is always reduced, while a complete hypergroup is never reduced. If we consider, given a hypergroup $H$, the grade fuzzy set $\widetilde{\mu}$ defined by Corsini [4, 6], then it is interesting to find relations between the notion of fuzzy grade of $H$ and that one of fuzzy reduced hypergroup. We aim also to obtain some relations between these two fuzzy approaches and the classical/crisp one.

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