THE THEORY OF HYPERVECTOR SPACES

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Abstract. This research concerns the strongly right and left distributive hypervector spaces, with examples and characterizations, the closure operator, the matroidal hypervector spaces, the hyperprojective and hyperaffine spaces associated with a hypervector space.

1. **Introduction**

Let K be a unitary ring, in particular a field. A *unitary K-hypermodule* is a quadruplet $(V, +, \circ, K)$, where $(V, +)$ is an Abelian group and

$$\circ : K \times V \to P'(V)$$

is a map of $K \times V$ into the set of non-empty subsets of V, such that (see [1], [3], [4]):

$$\forall\, a, b \in K,\ \forall\, x \in V \Rightarrow (a + b) \circ x \subseteq (a \circ x) + (b \circ x), \tag{1.1}$$

$$\forall\, a \in K,\ \forall\, x, y \in V \Rightarrow a \circ (x + y) \subseteq (a \circ x) + (b \circ x), \tag{1.2}$$

$$\forall\, a, b \in K,\ \forall\, x \in V \Rightarrow a \circ (b \circ x) = (ab) \circ x, \tag{1.3}$$

$$\forall\, a \in K,\ \forall\, x \in V \Rightarrow a \circ (-x) = (-a) \circ x = -(a \circ x), \tag{1.4}$$

$$\forall\, x \in V \Rightarrow x \in 1 \circ x. \tag{1.5}$$

If K is a field, the hypermodule $(V, +, \circ, K)$ is called *hypervector space* over K. (1.1) and (1.2) are called left and right distributivity respectively; (1.3) is called associativity. By (1.4) the two distributivities hold also for difference. A *K-hypermodule is strongly left distributive* if in (1.1) equality holds; *strongly right distributive*, if in (1.2) equality holds; if the same happens both in (1.1) and (1.2) the hypervector space is called simply *strongly distributive*. In the following, if not explicitly said, we assume that K is a field, so $(V, +, \circ, K)$ is a hypervector space.

A *subspace* of $(V, +, \circ, K)$ is a set $W \subseteq V$ such that

$$\begin{cases} W \neq \emptyset, \\ \forall\, x, y \in W \Rightarrow x - y \in W, \\ \forall\, a \in K,\ \forall\, x \in W \Rightarrow a \circ x \subseteq W. \end{cases} \tag{1.6}$$

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The quadruplet $(W,+,\circ,K)$ is obviously a hypervector space over K. We denote be S the family of all subspaces of V. Obviously it is:

$$\begin{cases} V \in S \\ \{W_i\}_{i\in I}, W_i \in S \Rightarrow \bigcap_{i\in I} W_i \in S\,. \end{cases} \tag{1.7}$$

Therefore S is a *closure system* in V. If $X \subseteq V$, the intersection of all subspaces containing X is a subspace $\overline{X}$, of V which is called the *closure* of X, or subspace spanned by X. It is the minimum subspace of V containing X. Then we define the *generators*, the *independents*, the *bases*, in the following way:

$$X \subseteq V, X \textit{ generator of } V \iff V = \overline{X}, \tag{1.8}$$

$$X \subseteq V, X \textit{ independent} \iff \forall\, x \in V, x \notin \overline{X - \{x\}}, \tag{1.9}$$

$$B \subseteq V, B \textit{ base } \text{of } V \iff B \textit{ independent and generator } \text{of } V \tag{1.10}$$

We say that V is finitely generated, if V has a finite basis.

In the following we set:

$$\Omega = 0 \circ \underline{0}, \tag{1.11}$$

where $\underline{0}$ is the zero element of $(V,+)$.

$$\forall\, x \in V,\ \gamma(x) = 0 \circ x, \tag{1.12}$$

$$\forall\, x \in V,\ U(x) = \bigcup_{a\in K} a \circ x\,. \tag{1.13}$$

We get:

$$\forall\, x \in V, \gamma(x) \subseteq U(x); x \in 1 \circ x \subseteq U(x)\,. \tag{1.14}$$

2. Strongly right distributive hypervector spaces

Let $(V,+,\circ,K)$ be a strongly right distributive hypervector space, that is such that in (1.2) equality holds. Then the same happens for the right distributivity for difference. We prove the following properties (see [1], sect. 2):

$$\forall\, a \in K, \quad \Omega = a \circ \underline{0} = a \circ \Omega \tag{2.1}$$

$$\forall\, x \in V, \quad \Omega \supseteq 0 \circ x = \gamma(x), \tag{2.2}$$

$$\Omega \quad \text{is a subgroup of} \quad (V,+), \tag{2.3}$$

$$|\Omega| = 1 \iff \forall\, a \in K,\ \forall\, x \in V, |a \circ x| = 1 \iff (V,+,\circ,K) \text{ is a classical vector space}, \tag{2.4}$$

$$\forall\, a \in K,\ \forall\, x \in V,\ \forall\ y \in a \circ x \Rightarrow y + \Omega = a \circ x\,. \tag{2.5}$$

By previous results we get:

Theorem 2.1. *Let $(V,+,\circ,K)$ be a Strongly right distributive hypervector space. The set $\Omega = 0\circ \underline{0}$ is a subgroup of $(V,+)$ and so the factor group $W = V/\Omega$ arises. For any $a \in K$ and $x \in V$, $\forall\, y \in a\circ x \Rightarrow [y] = y + \Omega = a \circ x$. Therefore the family $\{a\circ x : a \in K, x \in V\}$ coincides with the coset partition induced by the subgroup Ω in $(V,+)$. Inside the group $W = V/\Omega$ a classical product times a scalar in K is defined by setting:*

$$a\circ [x] = [y], \quad \textit{where} \quad y \in a\circ x\,.$$

In this way $(W,+,\circ,K)$, is a classical vector space. If $p : V \to W = V/\Omega$ is the canonical epimorphism between the groups V and W if we define:

$$\forall\, a \in K,\ \forall\, x \in V,\ a\circ x = p^{-1}(a \circ p(x))\,, \qquad (2.6)$$

the structure $(V,+,\circ,K)$ is a strongly distributive hypervector space.

By Theorem 2.1 immediately follows:

Theorem 2.2. *Every strongly right distributive hypervector space is strongly left distributive. That is it is strongly distributive.*

Theorem 2.2 completely characterizes the strongly distributive hypervector spaces. By Theorem 2.2 we get that the study of hypervector spaces must be continued by considering the strongly left distributive hypervector spaces, or, more generally, those not strongly distributive.

We remark that, as it is well known, $(Z,+)$ cannot be the additive group of a vector space over a field, however there are subgroups Ω of $(Z,+)$ such that Z/Ω is the additive group of a vector space over a field K. For instance we choose $\Omega = pZ$, p a prime and $K = Z_p$. It follows that Z is the additive group of a strongly distributive hypervector space over Z_p, where the hyperproduct times a scalar in Z_p is

$$[a] \in (Z_p,+), x \in (Z,+), [a]\circ x = \{ax + hp : h \in Z\}.$$

3. General results about strongly left distributive hypervector spaces

Let $(V,+,\circ,K)$ be a strongly left distributive hypervector space. We prove that the following properties hold (see [3]):

$$\Omega \textit{ is a subgroup of } (V,+), \qquad (3.1)$$

$$\forall\, a \in K,\ a\circ\underline{0} = \Omega,\ a\circ\Omega = \Omega, \qquad (3.2)$$

$$\forall\, x \in V,\ \gamma(x) = 0\circ x \supseteq \Omega, \qquad (3.3)$$

$$\forall\, a \in K,\ \forall\, x \in V, \gamma(x) = (a\circ x) - (a \circ x), \qquad (3.4)$$

$$\forall\, x \in V, \gamma(x) \textit{ is a subgroup of } (V,+) \textit{ containing } \Omega. \qquad (3.5)$$

In the following we denote by Γ the lattice of subgroups of $(V,+)$ containing Ω. By (3.5) the following map arises:

$$\gamma : x \in V \to \gamma(x) \in \Gamma \quad (\gamma(x) \supseteq \Omega), \qquad (3.6)$$

which enjoys the following properties:

$$\forall\, x \in V,\ \gamma(\gamma(x)) = \gamma(x), \tag{3.7}$$

$$\forall\, x, y \in V,\ \gamma(x+y) \subseteq \gamma(x) + \gamma(y), \tag{3.8}$$

$$\forall\, x \in V,\ \gamma(-x) = \gamma(x)\,. \tag{3.9}$$

By (3.7) it follows:

$$x \in \Omega \Rightarrow \gamma(x) = \Omega\,. \tag{3.10}$$

Moreover:

$$\forall\, x \in V,\ 1 \circ x = x + \gamma(x), \tag{3.11}$$

from which (for $x = 0$) we get:

$$1 \circ \underline{0} = \Omega. \tag{3.12}$$

We prove (see [3]):

Theorem 3.1. *If $x \in V \Rightarrow 0 \circ x = \Omega$, then V is strongly distributive and then it is completely characterized by Theorem* 2.1*, sect.* 2*.*

Therefore in the following we assume:

$$x \in V \quad \text{exists such that} \quad \gamma(x) = 0 \circ x \supset \Omega. \tag{3.13}$$

We prove:

$$x \in \gamma(x) \iff 1 \circ x = 0 \circ x = \gamma(x) \iff [\forall\, a \in K, a \circ x = \gamma(x)]. \tag{3.14}$$

By (3.13) and (3.9) it follows:

$$x \in \Omega \iff 1 \circ x = \gamma(x) = \Omega \iff [\forall\, a \in K, a \circ x = \Omega]. \tag{3.15}$$

In the following we set:

$$T = \{x \in V : 1 \circ x = \gamma(x)\} = \{x \in V : x \in \gamma(x)\}, \tag{3.16}$$

$$S = \{x \in V : x \in \gamma(x)\} = V - T, \tag{3.17}$$

$$R = \{x \in V : \gamma(x) = \Omega\}. \tag{3.18}$$

By (3.19), (3.13), (3.14) we get:

$$\Omega \subseteq R, \Omega \subseteq T, R - \Omega \subseteq S. \tag{3.19}$$

From now on, by (3.13), we assume $R \neq V$.

4. Characterization of strongly left distributive hypervector spaces, with $T = V$

Let $(V, +, \circ, K)$ be a hypervector space over the field K, which is *strongly left distributive, but not right*, such that $T = V$, that is

$$\forall\, x \in V,\ x \in \gamma(x). \tag{4.1}$$

It is $R \neq V$, otherwise, by Theorem 3.1, sect. 3, V were also right distributive, a case which we don't consider. Therfore:

$$\exists\, x \in V, \quad \textit{such that} \quad \gamma(x) \neq \Omega\,. \tag{4.2}$$

By (3.13) we get:

$$\forall\, a \in K,\ \forall\, x \in V,\ a \circ x = \gamma(x), \tag{4.3}$$

moreover the map γ, defined in (3.6) satisfies (3.7), (3.8), (3.9), (4.1), (4.2).

Let now $(V, +)$ be an Abelian group, Ω a subgroup and Γ the lattice of subgroups of $(V, +)$ through Ω. For any map:

$$\gamma : x \in V \to \gamma(x) \in \Gamma, \quad \text{with} \quad \gamma(\underline{0}) = \Omega, \tag{4.4}$$

satisfying (3.7), (3.8), (3.9), (4.1), (4.2) and for any field K, set:

$$\forall\, a \in K,\ \forall\, x \in V,\ a \circ x = \gamma(x). \tag{4.5}$$

We prove that $(V, +, \circ, K)$ is a strongly left, but not right distributive hypervector space, such that $T = V$. (1.1) is obvious, (1.2), (1.3), (1.4) follow by (3.8), (3.7), (3.9) respectively. (1.5) follows by (4.5) and (4.1), therefore $(V, +, \circ, K)$ is a strongly left distributive hypervector space such that $T = V$, by (4.1). Now we prove that it is not strongly right distributive. By (4.4) we get $\gamma(\underline{0}) = \Omega$; moreover by (4.2) an element $x \in V$ exists such that $\gamma(x) \neq \Omega$. Therefore by (3.9) we get:

$$\gamma(x) + \gamma(-x) = \gamma(x) \neq \Omega = \gamma(\underline{0}) = \gamma(x - x),$$

that is, setting $y = -x$ we have

$$\forall\, a \in K, (a \circ x) + (a \circ y) \neq a \circ (x + y).$$

Hence x and y in V exist such that in (1.2) equality doesn't hold. So we prove (see [3]):

Theorem 4.1. *Let $(V, +, \circ, K)$ be a strongly left, but not right distributive hypervector space, with $T = V$. The map γ defined in* (3.6) *satisfies* (3.7), (3.8), (3.9), (4.1), (4.2). *Conversely, let $(V, +)$ be any Abelian group, Ω a subgroup of $(V, +)$, Γ the lattice of subgroups of $(V, +)$ through Ω and γ a map* (4.4) *satisfying* (3.7), (3.8), (3.9), (4.1), (4.2). *Then for any field K, sety $a \circ x = \gamma(x)$, $\forall\, a \in K$, $\forall\, x \in V$. It follows that $(V, +, \circ, K)$ is a strongly left, but not right distributive hypervector space, with $T = V$.*

Theorem 4.1 completely characterizes the strongly left, but not right distributive hypervector spaces, with $T = V$. So the problem reduces to determine the groups $(V, +)$ endowed with a map (4.4) satisfying (3.7), (3.8), (3.9), (4.1), (4.2).

Let us now provide general examples of such a structure.

Example 4.1 Let $(V, +)$ be an Abelian group, Ω a subgroup of $(V, +)$, $\Omega \neq V$ and Γ the lattice of subgroups of $(V, +)$ through Ω. For any $x \in V$, let $\gamma(x)$ be the subgroup of $(V, +)$ spanned by x and Ω. Obviously it is $\gamma(\underline{0}) = \Omega$; moreover the map (4.4) just defined satisfies (3.7), (3.8), (3.9), (4.1), (4.2). Therefore, by Theorem 4.1, for any field K this map gives rise to a strongly left, and not right distributive hypervector space with $T = V$.

Example 4.2 Let $(V, +)$ be an Abelian group, Ω a subgroup of $(V, +)$, $\Omega \neq V$, Γ the lattice of subgroups of $(V, +)$ through Ω. Let $\{H_i\}_{i \in N}$ a family of subgroups of $(V, +)$ such that:

$$H_1 = \Omega,\ H_i \subseteq H_{i+1},\ \bigcup_{i \in N} H_i = V\,. \tag{4.6}$$

For any $x \in V$, let be $\gamma(x) = H_m$, where $m = \min\{i \in N : x \in H_i\}$. Then a map γ (4.4) arises and it is easy to check that it satisfies (3.7), (3.8), (3.9), (4.1), (4.2). By Theorem 4.1, this map determines a strongly left and not right distributive hypervector space over, with $T = V$, where K is any field.

5. The strongly left distributive hypervector spaces, with $T \neq V$

The results of sect. 4 are valid, more in general, for a strongly left distributive unitary K-hypermodule, where K is a whatever unitary ring. Now we assume that K is a field.

Let $(V, +, \circ, K)$ a hypervector space over the field K. Since K is a field, we prove (see [3]):

$$a, b \in K, x \in V, a \circ x = b \circ x \Rightarrow x \in \gamma(x), \text{ or } a = b. \tag{5.1}$$

By (5.1) and (3.16) it follows:

$$a \circ x = b \circ x, x \notin T \Rightarrow a = b. \tag{5.2}$$

By previous results, we prove:

$$(a \circ x) \cap (b \circ x) \neq \emptyset, a \neq b (a, b \in K, x \in U) \Rightarrow x \in T. \tag{5.3}$$

By (5.3), taking into account (3.16), (3.14), we get

$$\begin{aligned} \exists\, a, b \in K, a \neq b : (a \circ x) \cap (b \circ x) \neq \emptyset \iff x \in T \iff \\ \iff \forall\, c, d \in K, c \circ x = d \circ x. \end{aligned} \tag{5.4}$$

It follows:

$$\forall\, a, b \in K, a \neq b, x \in S = V - T \Rightarrow (a \circ x) \cap (b \circ x) = \emptyset. \tag{5.5}$$

By previous results we prove this following main condition:

$$x \in V, a \in K, y \in a \circ x \Rightarrow a \circ x = y + \gamma(x). \tag{5.6}$$

By (5.5) and (5.6) it follows:

Theorem 5.1. *For any $x \in V - T$ the family of subsets of $V \{a \circ x\}_{a \in K}$ is proper and consists of disjoint classes which are suitable cosets of $\gamma(x)$ in V.*

The following properties hold:

Theorem 5.2. *If $x \in S = V - T$, the set $U(x) = \bigcup_{a \in K} a \circ x$ is a subspace of $(V, +, \circ, K)$, with $U(x) \supseteq \gamma(x)$.*

Moreover:

$$y \in U(x) \Rightarrow U(y) \subseteq U(x). \tag{5.7}$$

Theorem 5.3. *If $T \neq V$ and V is finite, the field K is finite, that is, it is a Galois field $GF(q)$.*

Moreover for any $x \in S = V - T$, we get:

$$|\gamma(x)|q \quad \textit{divides} \quad |V|$$

whence q divides $|V|$. In particular, if V is finite, $T \neq V$ and $|V| = |K| = q$, then $(V, +, \circ, K)$ is the vector space $GF(q)$.

Theorem 5.4. *A strongly left distributive hypervector space V, with $T \neq V$, with prime order p, coincides with the vector space Z_p.*

6. A class of examples of strongly left and not right distributive hypervector spaces, with $T \neq V$

Let $(V, +, \circ, K)$ be a classical vector space over the field K and Γ the lattice of subspaces of V. Consider any *non-constant* map:

$$\gamma : x \in V \to \gamma(x) \in \Gamma, \tag{6.1}$$

satisfying to the following conditions:

$$\forall\, x \in V,\ \gamma(\gamma(x)) = \gamma(x), \tag{6.2}$$

$$\forall\, x \in V, \gamma(x + y) \subseteq \gamma(x) + \gamma(y), \tag{6.3}$$

$$\forall\, a \in K - \{0\}, \forall\, x \in V, \gamma(ax) = \gamma(x). \tag{6.4}$$

Set:

$$\Omega = \gamma(\underline{0}). \tag{6.5}$$

By (6.3) for $y = -x$ and by (6.4) for $a = -1$, we get:

$$\forall\, x \in V,\ \Omega \subseteq \gamma(x). \tag{6.6}$$

Moreover, by (6.2) we get:

$$x \in \gamma(\underline{0}) \Rightarrow \gamma(x) \subseteq \gamma(\gamma(\underline{0})) = \gamma(\underline{0}),$$

whence, by (6.6):

$$x \in \Omega \Rightarrow \gamma(x) = \Omega. \tag{6.7}$$

Since the map (6.1) is not constant, we have:

$$\exists\, x \in V,\ \text{such that } \gamma(x) \neq \Omega. \tag{6.8}$$

Set:

$$\forall\, a \in K, \forall\, x \in V, a \circ a = ax + \gamma(x). \tag{6.9}$$

We prove (see [3], sect. 4):

Theorem 6.1. $(V, +, \circ, K)$ *is a strongly left and not right distributive hypervector space, such that* $\gamma(x) = 0 \circ x$.

So the construction of such hypervector spaces depends only on not constant maps (6.1) *satisfying* (6.2), (6.3), (6.4).

We now provide some examples of such maps (6.1).

Example 6.1 Let $(V, +, \circ, K)$ be a classical vector space over a field K, Γ the lattice of its subgroups. Let $\{H_i\}_{i \in N}$ be a family of subspaces of V such that

$$\begin{cases} H_i \subseteq H_{i+1}, H = \bigcup_{i \in N} H_i \neq V, \\ \exists\, j \in N : H_j \neq H_{j+1}. \end{cases} \tag{6.10}$$

As H is a subspace of V, we set:

$$\begin{cases} x \in V - H, & \gamma(x) = V, \\ x \in H, & \gamma(x) = H_m, m = \min(i \in N : x \in H_i). \end{cases} \tag{6.11}$$

It is easy to prove that the map (6.1) as above is not constant and satisfies (6.2), (6.3), (6.4) (see [3], n. 4). By Theorem 6.1 then it follows that $(V, +, \circ, K)$ is a strongly left and not right distributive hypervector space. Moreover it is

$$T = \{x \in V : x \in \gamma(x)\} = H, S = V - H \neq \emptyset, R = \{x \in V : \gamma(x) = \Omega\} = \\ = \gamma(\underline{0}) = H_1 = \Omega.$$

Example 6.2 Let $(V, +, \circ, K)$ be a classical vector space over the field K, Ω a subspace of V, R and H two subspaces through Ω, such that H is not contained in R (and then $H \neq \Omega$ and $R \neq V$). Set:

$$\begin{cases} x \in R, & \gamma(x) = \Omega, \\ x \in V - R, & \gamma(x) = H. \end{cases} \tag{6.12}$$

The map (6.1) as above is not constant and we can easily check that it satisfies (6.2), (6.3), (6.4). By Theorem 6.1 it follows that $(V, +, \circ, K)$ is a strongly left and not right distributive hypervector space. Moreover it is:

$$S = \{x \in V : x \notin \gamma(x)\} = (V - H) \cap (R - \Omega)$$

and them if $H \neq V$, or if $R \neq \Omega$, we have $S \neq \emptyset$, that is $V \neq T$. At last we get:

$$R = \{x \in V : \gamma(x) = \Omega\}.$$

Example 6.3 Let $(V,+,\circ,K)$ a classical vector space over K; L and H two subspaces such that $L \neq (\underline{0})$ and $L \subseteq H$. Set:

$$\begin{cases} x \in L, & \gamma(x) = \text{ space spanned by } x, \\ x \in V - L, & \gamma(x) = H. \end{cases} \tag{6.13}$$

The map (6.1) as above is not constant and satisfies to (6.2), (6.3), (6.4). Therefore, as in (6.9), we get a strongly left, but not right distributive hypervector space $(V,+,\circ,K)$.

This hypervector space is such that $T = H$, $S = V - H$, $\Omega = R = (\underline{0})$.

7. Some properties of the closure operator in a hypervector space

From here onward we assume that $(V,+,\circ,K)$ is a hypervector space over the field K not necessarily either left, or right strongly distributive. First of all, let us remark that there are hypervector spaces which are not strongly distributive either right, or left.

Example 7.1 Let $(V,+,\circ,\mathbb{R})$ a classical vector space over the reals $\mathbb{R}$. For any $x \in V$ set:

(7.1) $a \circ x = [0, ax] =$the set of vectors through the origin and end point belonging to the closed segment $[\circ, ax]$.

We prove that $(V,+,\circ,\mathbb{R})$ as in (7.1) is a hypervector space over the reals, which is not strongly distributive either left or right. This hypervector space (see (1.11), (1.12), (1.13)) is such that:

(7.2) $\Omega = 0 \circ \underline{0} = (\underline{0})$,

(7.3) $\forall\, x \in V, \gamma(x) = 0 \circ x = (\underline{0}) = \Omega$,

(7.4) $\forall\, x \in V, U(x) = \bigcup_{a \in K} a \circ x = (x) =$ space spanned by x in $(V,+,\circ,K)$

In this section we deal with the closure operator in any hypervector space $(V,+,\circ,K)$ (see sect.1). By (1.6) we get:

$$\forall\, W \in S,\ \forall\, x \in W,\ \forall\, a \in K \Rightarrow a \circ x \subseteq W. \tag{7.5}$$

It follows by (1.13):

$$\forall\, x \in V,\ U(x) \subseteq \{\overline{x}\}. \tag{7.6}$$

Moreover (see (1.3), (1.11), (1.12)):

$$\forall\, x \in V : \underline{0} \in \gamma(x) \Rightarrow \Omega \subseteq \gamma(x), \tag{7.7}$$

$$\forall\, x \in V : \underline{0} \in U(x) \Rightarrow \Omega \subseteq \gamma(x). \tag{7.8}$$

By (7.7), (7.8) and (1.14) it follows:

Theorem 7.1. *If* $0 \in \Omega$*, then* $\forall\, x \in V$*:*

$$0 \in \gamma(x) \iff \Omega \subseteq \gamma(x) \iff \underline{0} \in U(x) \iff \Omega \subseteq U(x). \tag{7.9}$$

By (7.6) *it follows that:*

$$\Omega \subseteq \overline{\{\underline{0}\}}. \tag{7.10}$$

Since every subspace $W \in S$ contains $\underline{0}$, we have that $\Omega = 0 \circ \underline{0} \subseteq W$.
It follows:

$$\Omega \subseteq \overline{\emptyset}. \tag{7.11}$$

We prove (see [4], sect. 2):

$$\Omega \in S \iff \Omega - \Omega \subseteq \Omega \iff (\Omega, +) \subseteq (V, +), \tag{7.12}$$

$$\forall x \in V, U(x) = \{\overline{x}\} \iff U(x) - U(x) \subseteq U(x) \iff (U(x), +) \subseteq (V, +). \tag{7.13}$$

By (7.11) and (7.12) we get:

$$(\Omega, +) \subseteq (V, +) \iff \Omega = \overline{\emptyset}. \tag{7.14}$$

We prove (see [4], sect. 2):

$$W, W' \in S \Rightarrow W + W' = \overline{W \cup W'}. \tag{7.15}$$

If W, W' are two subspaces, by previous results we relate to them the spaces *intersection* and *span*, $W \cap W'$ and $W + W' = \overline{W \cup W'}$, respectively.

The notion of span as in (7.15) generlizes obviously to any number of subspaces. We prove (see [4], sect. 2):

Theorem 7.2. *Let* $x_1, x_2, \ldots, x_n$ *be elements of* V *such that* $U(x_i)$ *is a subgroup of* $(V, +)$*,* $i = 1, 2, \ldots, n$*. Then it is*

$$\{\overline{x_1, x_2, \ldots, x_n}\} = U(x_1) + U(x_2) + \ldots + U(x_n). \tag{7.16}$$

At last we remark:

$$x \in V, x \in \gamma(x) \Rightarrow U(x) \subseteq \gamma(x). \tag{7.17}$$

A hypervector space is called *regular*, if:

$$\{\forall\, x \in V, \gamma(x) \in S, U(x) \in S\}. \tag{7.18}$$

The previous Example 7.1 is a regular hypervector space. Moreover every strongly left distributive hypervector space is regular and therefore every strongly distributive hypervector space is regular.

By (7.18) we get:

$$V \ \textit{regular} \begin{cases} \forall\, x \in V, \underline{0} \in \Omega \subseteq \gamma(x), \\ \Omega = \overline{\emptyset}, \\ \forall\, x \in \Omega \Rightarrow U(x) = \Omega. \end{cases}$$

If V is regular and finitely generated there are $x_1, x_2, \ldots, x_n \in V$ such that (see Theorem 7.2):

$$\forall\, x \in V,\ \exists\, a_1, a_2, \ldots, a_n \in K : x \in (a_1 \circ x_1) + (a_2 \circ x_2) + \ldots + (a_n \circ x_n). \tag{7.19}$$

8. Matroidal hypervector spaces

A hypervector space $V = (V, +, \circ, K)$ is called *matroidal* if the closure operator (see sect. 1) satisfies the following exchange axiom (see [5]):

$$\forall x, y \in V, \forall X \subseteq V : x \notin \overline{X}, x \in \overline{X \cup \{y\}} \Rightarrow y \in \overline{X \cup \{x\}}. \tag{8.1}$$

We say that U is n-matroidal ($n \in N$) if (8.1) holds for every subset X of V, with $|X| \leq n$.

In the following we assume V finitely generated. The following theorems hold (see [5]).

Theorem 8.1. *If in V the set X is independent and $X \cup \{y\}$ is dependent, then $y \in \overline{X}$. If V is regular, we have (see Theorem 7.2, sect. 7):*

$$\begin{aligned} &\big[X = \{x_1, x_2, \dots, x_n\} \text{ independent, } X \cup \{y\} \text{ dependent } \big] \Rightarrow \\ &\Rightarrow \exists a_1, a_2, \dots, a_n \in K : y \in (a_1 \circ x_1) + (a_2 \circ x_2) + \ldots + (a_n \circ x_n). \end{aligned} \tag{8.2}$$

Theorem 8.2. *For any $W \in S$, two bases of V have the same size, called dimension,* $\dim W$*, of W. We have*

$$W, W' \in S \Rightarrow \dim W + \dim W' \geq \dim(W \cap W') + \dim(\overline{W \cup W'}). \tag{8.3}$$

Moreover, since

$$\{x\} \text{ independent } \Rightarrow x \notin \overline{\emptyset}, \tag{8.4}$$

$\overline{\emptyset}$ has no independents and therefore every basis is empty, that is:

$$\dim \overline{\emptyset} = 0\,. \tag{8.5}$$

We prove (see [4], sect. 3):

Theorem 8.3. *If V is 0-matroidal, then for any $x, y \in V - \overline{\emptyset}$ we have:*

$$y \in \{\overline{x}\} \iff \{\overline{y}\} = \{\overline{x}\}. \tag{8.6}$$

It follows:

$$x, y \in V, \{\overline{x}\} \neq \{\overline{y}\} \Rightarrow \{\overline{x}\} \cap \{\overline{y}\} = \overline{\emptyset}. \tag{8.7}$$

Theorem 8.4. *If V is 1-matroidal, we get:*

$$x, y \in V - \overline{\emptyset}, u, v \in \{\overline{x, y}\} - \overline{\emptyset}, u \notin \{\overline{v}\} \Rightarrow \{\overline{u, v}\} = \{\overline{x, y}\}. \tag{8.8}$$

Theorem 8.5. *If V is regular and 0-matroidal, we get: $V = T \cup R$, $T \cap R = \overline{\emptyset}$, where T and R are defined in* (3.16) *and* (3.18) *respectively.*

9. Hyperprojective space associated with a 1-matroidal hypervector space

Let $V = (V, +, \circ, K)$ be a 1-matroidal hypervector space. We define in $V - \overline{\emptyset}$ the following relation ρ:

$$x, y \in V - \overline{\emptyset}, x \rho y \iff y \in \{\overline{x}\}. \tag{9.9}$$

The above relation is reflexive by (1.14) and (1.6); symmetric by (1.14), (7.6) and (8.6) and it is obviously transitive.

Therefore ρ is an equivalence in $V - \overline{\emptyset}$. The equivalence class of $x \in V - \overline{\emptyset}$ is $\{\overline{x}\} - \overline{\emptyset}$. Set

$$\vec{x} = \{\overline{x}\} - \overline{\emptyset}. \tag{9.10}$$

So we get the factor set:

$$\mathbb{P}(V) = (V - \overline{\emptyset})/\rho \tag{9.11}$$

and the canonical projection

$$p : x \in V - \overline{\emptyset} \to p(x) = \vec{x} \in \mathbb{P}(V). \tag{9.12}$$

If $W \in S$ the projection of $W - \overline{\emptyset}$ is called subspace of $\mathbb{P}(V)$. The family of subspaces of $\mathbb{P}(V)$ is denoted by $\mathcal{P}$. The pair $(\mathbb{P}, \mathcal{P})$ is a closure space, called hyperprojective space associated with V.

Obviously $(\mathbb{P}, \mathcal{P})$ satisfies the exchange axiom, if V is matroidal.

If $P \in \mathcal{P}$, it is $P = p(W - \overline{\emptyset})$, with $W \in S$. If W is finitely generated, P in $(\mathbb{P}, \mathcal{P})$ is finitely generated too.

The 1-dimensional subspaces of $(\mathbb{P}, \mathcal{P})$ are called lines and if $\mathcal{L}$ is the family of lines of $(\mathbb{P}, \mathcal{P})$, it is:

Theorem 9.1. $(\mathbb{P}, \mathcal{L})$ *is a linear space, in which the line through* $\vec{x}$ *and* $\vec{y}$ $(\vec{x} \neq \vec{y})$ *is* $\{\overline{x, y}\} - \emptyset$. *This line is unique by* (8.8).

We remark that in general a hyperprojective space, although is a linear space, is not a projective space, since it is not true that the closure of three independent points is a projective plane.

If V is matroidal, then:

$$W \in S, P = p(W - \overline{\emptyset}) \in \mathcal{P}, \dim P = \dim W - 1, \tag{9.13}$$

where $\dim P$ is the size of one, and there of every basis of P in $(\mathbb{P}, \mathcal{P})$.

By (9.13) we get (see (8.5)):

$$\dim \emptyset = -1, \dim \vec{x} = 0. \tag{9.14}$$

10. **Hyperaffine space associated with a regular hypervector space**

Let $V = (V, +, \circ, K)$ be a regular hypervector space (see (7.18)). Denote by $\mathcal{R}$ the family of cosets of the subgroups $U(x)$, $x \notin \Omega$ in $(V, +)$, that is if we set:

$$\forall x, y \in V, x \notin \Omega, r(x, y) = y + U(x), \tag{10.1}$$

it is

$$\mathcal{R} = \{r(x, y) : x, y \in V, x \notin \Omega\}. \tag{10.2}$$

The geometric space $(V, \mathcal{R})$ is called *hyperaffine space* associated with V. We prove (see [4], sect. 3):

$$\forall x, y \in V \to \exists r \in \mathcal{R} : x, y \in r. \tag{10.3}$$

We call *lines* the elements of $\mathcal{R}$. In $(V,\mathcal{R})$ a parallelism relation between lines is defined in the following way:

$$r(x,y)\|r(x',y') \iff U(x)=U(x'). \tag{10.4}$$

This relation is an equivalence and satisfies the *Euclid axiom*:

$$\forall x\in V, \forall r\in\mathcal{R} \Rightarrow \exists! r'\in\mathcal{R} : x\in r', r\|r'. \tag{10.5}$$

We remark that two distinct lines meet in at most one point and therefore through two distinct points there is *not a unique line*. More precisely we have:

$$\begin{aligned} &r_1=r(x_1,y_1), r_2=r(x_2,y_2)\in\mathcal{R},\\ &z\in r_1\cap r_2 \Rightarrow r_1\cap r_2 = z+U(x_1)\cap U(x_2). \end{aligned} \tag{10.6}$$

Let us assume that V is 0-matroidal and regular. Then by Theorem 8.2 of sect. 8 and since $\overline{\emptyset}=\Omega$, we get:

$$\begin{aligned} &\forall x_1,x_2\in V-\Omega, U(x_1)=U(x_2),\\ &\text{or } U(x_1)\cap U(x_2)=\Omega. \end{aligned} \tag{10.7}$$

Therefore, if $\Omega=(\underline{0})$, by (10.6) and (10.7) the space $(V,\mathcal{R})$ is a linear space where a parallelism is defined.

It follows:

Theorem 10.1. *If $V=(V,+,\circ,K)$ is a regular and 0-matroidal hypervector space, such that $\Omega=(\underline{0})$, the hyperaffine space associated with $(V,\mathcal{R})$ is a linear space endowed with a parallelism. If all lines have the same size, it is therefore a Sperner space.*

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