AN ANALYSIS OF SOCIAL RELATIONS AND SOCIAL GROUP BEHAVIORS WITH FUZZY SETS AND HYPERSTRUCTURES

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Abstract. Starting by the modellization of social relations by Moreno, having as mathematical tools binary matrices and graph theory, we propose some new models based on algebraic hyperstructures, fuzzy sets and fuzzy hyperstructures. One of the main ideas is that the social aggregations can be described as algebraic hyperoperations. Another idea is that an algebraic hyperoperation, having many possible results, is useful to represent uncertainty on the result of an aggregation between individuals. Moreover, the relations considered by Moreno can be usefully replaced by fuzzy relations that are more precise in description of human relationships.

1. Sociological indices in a Social Group

1.1. **The Moreno indices.** The modellization of the Social Relations starts with a survey technique developed by Moreno in 1946 and published in [16] in 1953. Many other authors have followed the Moreno indications. The mathematical tools used by Moreno et al. [15, 16] is the theory of relations in a set $U = \{x_1, x_2, \dots, x_n\}$ of individuals that form a social group (e.g. the students in a school, the professors in a university, etc.). Every relation R is represented by a matrix $M_R = (m_{rs})$ where $m_{rs} = 1$ if $x_r \, R \, x_s$ and $m_{rs} = 0$ if x_r is not in the relation R with x_s. To highlight the sociological significance M_R it is called *socio-matrix* associated with the social group U and the relation R. Many sociological results follow from the distribution of the number 1 in the socio-matrix. Let us illustrate the Moreno ideas by an example. Let us write $x_r \, R \, x_s$ if the individual x_r *likes* the individual x_s in a *particular activity* A. Individual x_r can choose from 1 to $n-1$ individuals x_s for the collaboration, and similarly $n-1$ individuals x_r can choose x_s. At least an

Received: 17 September 2015 2015 ; Accepted: 14 December 2015
Key words and phrases: Social Relations, Social Aggregations, Hyperoperations, Fuzzy Structures, Mathematical Models for Social Sciences

individual must be chosen. Let c_r^1 be the number of the persons x_s chosen by x_r for the activity A. The quotient

$$\gamma_r^1 = c_r^1/(n-1)$$

is a number belonging to the interval $[0, 1]$ and represent the degree in which x_r trust the other members of the group, and then he feels integrated with them. Then γ_r^1 is said to be the *integration index* of x_r. Moreover, let d_s^1 the number of the individuals x_r that choose x_s for the activity A.
The quotient

$$\delta_s^1 = d_s^1/(n-1)$$

is a number belonging to the interval $[0, 1]$ and represent the degree in which the other members of the group trust x_r. Then δ_s^1 is a measure of the prestige of x_s and is called the *prestige index* of x_s.

1.2. **The negative and indifference indices.** From a crisp viewpoint, the position of an individual x_r over another individual x_s, in relation to an activity A can be of 3 types:

(1) x_r choose x_s;
(2) x_r reject x_s;
(3) x_r does not choose and does not reject x_s.

Then we can introduce the following "negative indices".
Let c_r^{**1} be the number of individual x_s refused by x_r for cooperation (e.g. individuals unpleasant, disagreeable, unsuitable for cooperation, etc.) and d_s^{**1} the number of individuals x_r that refuse x_s. Then we can consider two "negative" indices:

- the *anti-integration index* defined as:

$$\gamma_r^{**1} = c_r^{**1}/(n-1)$$

- the *anti-prestige index*

$$\delta_s^{**1} = d_s^{**1}/(n-1).$$

Finally, let c_r^{*1} be the number of individual x_s not chosen and not refused by x_r for cooperation, and let δ_s^{*1} be the number of individuals x_r that not choose and not refuse x_s. We can introduce two "indifference" indices:

- the *weak-integration index* defined as:

$$\gamma_r^{*1} = c_r^{*1}/(n-1)$$

- the *weak-prestige index* given by:

$$\delta_s^{*1}/(n-1).$$

Remark 1.1. We agreed that the number of stars denotes the degree of negativity of an index.

Remark 1.2. Evidently, the following formulas hold:

$$\gamma_r^1 + \gamma_r^{*1} + \gamma_r^{**1} = 1; \qquad \delta_s^1 + \delta_s^{*1} + \delta_s^{**1} = 1 \tag{1}$$

$$\sum_r \gamma_r^1 = \sum_s \delta_s^1; \quad \sum_r \gamma_r^{*1} = \sum_s \delta_s^{*1}; \quad \sum_r \gamma_r^{**1} = \sum_s \delta_s^{**1}.$$

1.3. **The m-steps indices.** Generalizing the concepts introduced in Sections 1.1,1.2, we say that x_r *like in m-steps* x_s if:

(a) there is a finite sequence of $m+1$ individuals, $y_1 = x_r, y_2, \ldots, y_{m+1} = x_s$ such that $y_i \neq y_j$, for every i, j in $\{1, 2, m+1\}$, and y_i chooses y_{i+1};

(b) if $k < m$, then x_r do not like in k steps x_s.

Let us denote with c_r^m the number of persons x_s such that x_r like in m-steps x_s. We can define the *m-step integration index* of the individual x_r the number:

$$\gamma_r^m = c_r^m/(n-1).$$

Remark 1.3. Evidently, for $m > n-1$, $c_r^m = 0$, and $c_r^0 = \sum_m c_r^m \leq n-1$.

We define the *potential* (or *global*) *integration index* of the individual x_r the number:

$$\gamma_r^0 = c_r^0/(n-1).$$

Similarly d_s^m is the number of persons x_r such that x_r like in m steps x_s and we define the m-steps prestige index of an individual x_s as the quotient:

$$\delta_s^m = d_s^m/(n-1).$$

Remark 1.4. Evidently, for $m > n-1$, $d_r^m = 0$, and $d_r^0 = \sum_m d_r^m \leq n-1$.

Finally we define the *potential* (or *global*) *prestige index* of the individual x_r the number:

$$\delta_r^0 = d_r^0/(n-1).$$

Similarly we can define the m-steps negative and indifference indices.

2. Geometric spaces and hyperstructures associated to a socio-matrix

Let us give the following definition.

Definition 2.1. For every positive integer $m < n$ the *geometric space* of order m associated to the socio-matrix M is the pair $G^m = (S^m, d)$, where S^m is the set of the ordered quadruplet of real numbers $(\gamma_r^m, \delta_r^m, \gamma_r^{m**}, \delta_r^{m**})$, $r \in \{1, 2, \ldots, n\}$, and d is a metric on S^m.

In the sequel we limit ourselves to the case where $m = 1$ and d is the restriction to S of the Euclidean metric on R^4. In this case we do not write the superscript 1 and then we write G, S, $(\gamma_r, \delta_r, \gamma_r^{**}, \delta_r^{**})$ to denote G^1, S^1, $(\gamma_r^1, \delta_r^1, \gamma_r^{1**}, \delta_r^{1**})$, respectively. The ordered quadruplet $(\gamma_r, \delta_r, \gamma_r^{**}, \delta_r^{**})$ represents the individual x_r and, in this contest, of the quadruplets are all distinct, we can write $x_r = (\gamma_r, \delta_r, \gamma_r^{**}, \delta_r^{**})$, and so $U = S$.

Remark 2.2. From formula (1) it follows that it is not necessary to consider the indices γ_r^*, δ_r^*, for the characterization of x_r as they are a consequence of the other indices.

Let us denote with $[a, b]$ the usual closed interval $[a, b]$ if $a \leq b$, and the closed interval $[b, a]$ if $a > b$. Moreover, let us denote with $x_r x_s$ (or with $gs(x_r, x_s)$) the *geometric segment* with extremes x_r and x_s, i.e.

$$\{x_k \in R^4 \colon \exists \lambda \in [0,1] | x_k = \lambda x_r + (1-\lambda) x_s\},$$

and with $[x_r, x_s]$ (or with $gi(x_r, x_s)$) the *geometric interval*, i.e.

$$\{x_k = (x_k^1, \ldots, x_k^4) \in R^4 \colon x_k^i \in [x_r^i, x_s^i], i = 1, \ldots, 4\}.$$

We suppose that the reader is familiar with the terminology and notation used in hyperstructure theory. We use e.g. [1, 2, 3, 10] for terminology and notation which are not define here. The following facts are some definitions and propositions in the theory of hyperstructure which we need for formulation of our ideas and results.

A *hypergroupoid* is a pair $(H, \cdot)$ where H is a (nonempty) set and $\cdot \colon H \times H \to \mathcal{P}^*(H) \ \ (= \mathcal{P}(H) \setminus \{\emptyset\})$ is a binary hyperoperation on the set H.

If $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ for all $a, b, c \in H$, (associativity), then $(H, \cdot)$ is called a *semihypergroup*.

A semihypergroup $(H, \cdot)$ is said to be a *hypergroup* if the reproduction axiom $a \cdot H = H = H \cdot a$ for all a $\in H$, is satisfied.

We define the following hyperproducts associated to S:

(a) the *social segment* $s(x_r, x_s) = x_r\, s\, x_s$, as the set

$$\{x_k \in S : \exists \lambda \in [0, 1] | x_k = \lambda x_r + (1 - \lambda) x_s\};$$

(b) the *social interval* $i(x_r, x_s) = x_r\, i\, x_s$, as the set

$$\big\{x_k = (\gamma_k, \delta_k, \gamma_k^{**}, \delta_k^{**}) \in S \colon \\ \gamma_k \in [\gamma_r, \gamma_s], \delta_k \in [\delta_r, \delta_s], \gamma_k^{**} \in [\gamma_r^{**}, \gamma_s^{**}], \delta_k^{**} \in [\delta^{**} r, \delta_s^{**}]\big\}.$$

The hyperstructures (S, s) and (S, i) have the following properties:

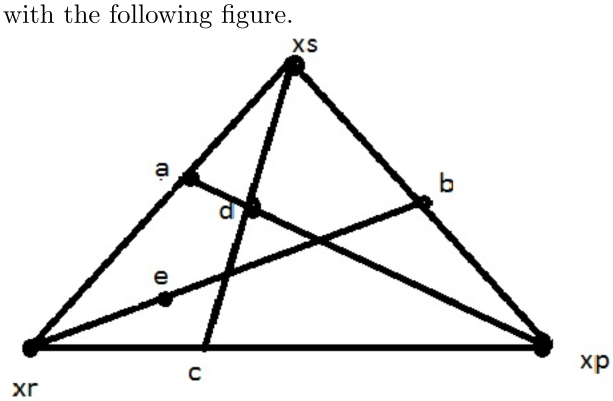
(P1) idempotence;

(P2) commutativity;

(P3) quasi hypergroups.

So, they are geometric hyperstructures.

But, unlike the analogous hyperstructures defined in R^4, they are not associative, as it is proved with the following figure.



We show a case in which

$$(x_r\, s\, x_s)\, s\, x_q = \{x_r, x_s, x_q, a, b, c, d\}, \text{ while}$$
$$x_r\, s\, (x_s\, s\, x_q) = \{x_r, x_s, x_q, a, b, c, e\}.$$

The lack of associativity can appear strange to those accustomed to working with join spaces. In fact, if we define as hyperproducts the geometric segment $x_r x_s$ or the geometric interval $[x_r, x_s]$, it is well known that the associative property holds. The apparent abnormality depends on the fact that here we are in a social context and not in a theoretical/geometrical context. Then previous hyperproducts define social coalitions, that in general are not associative.

The hyperproduct $x_r \, s \, x_s$ has the meaning of a social coalition between x_r and x_s, and $(x_r \, s \, x_s) s x_q$ is the union of the social coalitions between every element of $x_r \, s \, x_s$ and x_p. The individual "e" is not in the union of these coalitions. Similarly $x_r \, s \, (x_s \, s \, x_q)$ is the union of the coalitions between x_r and every element of $x_s \, s \, x_q$, then d is not in such coalition.

The hyperproduct $s(x_r, x_s)$ can be too restrictive in a social context, as we may be thinking that an agreement between two individuals can attract not only people of the geometric segment $x_r x_s$, but also other nearby. Then, with the idea of enlarge the segment keeping idempotency and commutativity, we can consider the social interval $i(x_r, x_s)$.

It may be noted, however, that considering the metric d in S, the social interval $i(x_r, x_s)$ can contain individuals too far away not only from x_r and x_s but also by the geometric segment with extremes x_r and x_s. Then some possible alternatives can be appropriate, dependent on the social context of reference.

In this order of ideas we propose the following hyperproduct, with ε a suitable positive real number and $\bar{\varepsilon}$ is any vector perpendicular to the geometric segment x_r, x_s and with length ε .

Definition 2.3. Let ε be a nonnegative real number. The *geometric ε-segment* with extremes x_r and x_s is the set

$$gs_\varepsilon(x_r, x_s) = \{x_k \in R^4 \colon \lambda \in [0,1], \exists \varepsilon \perp gs(x_r, x_s) | x_k \in gs(\lambda x_r + (1-\lambda) x_s, \\ \lambda x_r + (1-\lambda) x_s + \lambda(1-\lambda)\bar{\varepsilon})\}.$$

The *social ε-segment* with extremes x_r and x_s is the set

$$s_\varepsilon(x_r, x_s) = gs_\varepsilon(x_r, x_s) \cap S, \quad \text{i.e.}$$

$$s_\varepsilon(x_r, x_s) = \{x_k \in S \colon \lambda \in [0,1], \exists \bar{\varepsilon} \perp gs(x_r, x_s) \\ | x_k \in gs(\lambda x_r + (1-\lambda) x_s, \lambda x_r + (1-\lambda) x_s + \lambda(1-\lambda)\bar{\varepsilon})\}.$$

Then we can define the hyperproduct:

$$s_\varepsilon : (x_r, x_s) \in S^2 \to x_r \, s_\varepsilon \, x_s = s_\varepsilon(x_r, x_s).$$

Corollary 2.4. *The hypergroupoid (S, s_ε) is commutative, idempotent and a quasi-hypergroup. Moreover it is weak associative. For $\varepsilon = 0, s_\varepsilon(x_r, x_s)$ reduces to $s(x_r, x_s)$.*

3. **Group indices and associate hyperstructures**

3.1. **The group indices.** Some further indices are concerning the whole group U. Let $n(R)$ be the number of the pairs (x_r, x_s), $r \neq s$, belonging to the relation R.

In (Moreno, 1953) the quotient

$$k(R) = n(R)/(n(n-1))$$

is considered, called the *total cohesion* of the group U.

The *symmetric part* of R is the relation $R_\sigma = R \cap R^{-1}$, the *anti-symmetric part* of R is $R_\alpha = R - R_\sigma$. The numbers

$$k(R_\sigma) = n(R_\sigma)/(n(n-1)), \qquad k(R_\alpha) = n(R_\alpha)/(n(n-1))$$

are, respectively, the *symmetric* and *anti-symmetric cohesion* of the group U and their sum is $k(R)$.

3.2. **Hyperstructures associated to group indices.** Starting by the cohesion indices of a group U, we can calculate the cohesion indices of the subgroups of U with at least two individuals. Let us introduce two suitable numbers $a, b \in [0, 1]$, $a \leq b$, called respectively, the *minimum* and the *maximum level of cohesion* (appropriate for a particular social context).

Then we can divide the subgroups of U in three categories:

- *subgroups with strong cohesion*, having cohesion index greater than b;
- *subgroups with sufficient cohesion*, if the cohesion index belongs to the interval $[a, b]$.
- *subgroups with weak cohesion*, with cohesion index less than a;

For every pair (x_r, x_s) of elements of U and for every interval $H = [a, b] \subseteq [0, 1]$ let us introduce the following (possible partial) hyperoperations associated to $[a, b]$:

- σ_H, where $x_r \, \sigma_H \, x_s$ is the union of the subgroups containing $\{x_r, x_s\}$ and with cohesion index belonging to H;
- σ_H^+, where $x_r \, \sigma_H^+ \, x_s$ is the union of the subgroups containing $\{x_r, x_s\}$ and with cohesion index greater than b;
- σ_H^-, where $x_r \, \sigma_H^- \, x_s$ is the union of the subgroups containing $\{x_r, x_s\}$ and with cohesion index less than a.

From a more restrictive point of view, we can introduce also the following (possible partial) hyperoperations associated to $[a, b]$:

- δ_H, where $x_r \, \delta_H \, x_s$ is the intersection of the maximal subgroups containing $\{x_r, x_s\}$ and with cohesion index belonging H.
- δ_H^+, where $x_r \, \delta_H^+ \, x_s$ is the intersection of the maximal subgroups containing $\{x_r, x_s\}$ and with cohesion index greater than b.
- δ_H^-, where $x_r \, \delta_H^- \, x_s$ is the intersection of the maximal subgroups containing $\{x_r, x_s\}$ and less than a.

The properties of the previous hyperoperations can give a very useful information on the cohesion properties of the social group U. In particular, the hypergroupoids connected to symmetric cohesion indices give information on the *democracy* (or *friendliness*) in the group. Moreover, the ones associated to antisymmetric cohesion indices individuate the mechanism of formation of *hierarchical groups*.

4. Sociological fuzzy indices and fuzzy hyperoperations

Another generalization of the Moreno models is obtained by considering fuzzy relations, that appear to be more adequate than the crisp one to represent human perceptions and communication. Every fuzzy relation R in the set $U = \{x_1, x_2, \ldots, x_n\}$ of individuals is represented by a fuzzy socio-matrix $M_R = (m_{rs})$ where $m_{rs} \in [0,1]$ is the degree in which the fuzzy relation R holds. We write $x_r\, R\; x_s = m_{rs}$. The sociological results known for the crisp socio-matrices can be, in many ways, extended to fuzzy socio-matrices.

4.1. **Individual fuzzy indices.** Let us illustrate such possible extension by an example. For $r \neq s$, let $x_r\, R\; x_s$ represent the degree in which the individual x_r likes the individual x_s in a particular activity A.

Let us denote with ${c_r}^1$ the sum of the numbers $x_r\, R\; x_s$ for $r \neq s$, r fixed, i.e. the sum of the degrees in which the persons x_s are chosen by x_r. Morever, we denote with ${d_s}^1$ the sum of the numbers $x_r\, R\; x_s$ for $r \neq s$, s fixed. The quotients

$$ {\gamma_r}^1 = {c_r}^1/(n-1), \qquad {\delta_s}^1 = {d_s}^1/(n-1) $$

are, respectively, the *fuzzy integration index* of x_r and the *fuzzy prestige index* of x_s.
Similarly we can introduce the "negative fuzzy evaluation criteria". We introduce a fuzzy relation R^{**} of "refusing". For the extension to the fuzzy ambit of the properties considered in the previous sections we assume the property

$$ x_r\, R\; x_s + x_r\, R^{**}\; x_s \leq 1. $$

We denote with $c_s^{**\ 1}$ the sum of the numbers $x_r\, R^{**}\, x_s$ for $r \neq s$, r fixed, i.e. the sum of the degrees in which the individuals x_s are refused by x_r for cooperation. Analogously we can define d_s^{**1} as the sum of the numbers $x_r\, R^{**}\, x_s$ for $r \neq s$, s fixed. Then we have two "negative" fuzzy indices:

$$ \gamma_r^{**1} = c_r^{**1}/(n-1), \delta_s^{**1} = d_s^{**1}/(n-1) $$

Let us call them *fuzzy anti-integration index* and *fuzzy anti-prestige index*, respectively.

4.2. **The fuzzy group indices.** Other fuzzy indices are concerning the entire group U. Let $n(R\ )$ be the sum of all the numbers $x_r\, R\; x_s$, $r \neq s$. The quotient

$$ k(R\ ) = n(R\ )/(n(n-1)) $$

is the total fuzzy cohesion of the group U.

The *symmetric part* of R is the relation $R_\sigma = R\ \cap R^{\ -1}$, the *anti-symmetric part* of R is $R_\alpha = R\ - R_\sigma$. The numbers $k(R_\sigma)$ and $k(R_\alpha)$ are, respectively, the *symmetric* and *anti-symmetric fuzzy cohesion* of the group U and their sum is $k(R\ )$.

Conclusions and research perspectives

Starting from the fuzzy indices considered in Sec. 4 we can extend all the definitions on the algebraic hyperstructures considered in Sec. 2 and in Sec. 3 to the fuzzy context. The aims of the further research program are:

- to describe the social meanings of the crisp and fuzzy indices and associated hyperstructures;
- to modellize social phenomena with new crisp and fuzzy indices and new hyperstructures.

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