HOMOMORPHISMS OF EL–HYPERSTRUCTURES BASED ON A CERTAIN CLASSICAL TRANSFORMATION

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Abstract. Motivated by properties of the Laplace transformation and certain types of ordinary differential equations a number of single-valued structures of operators of specific types has so far been constructed. Also, hyperstructures of a certain type have been constructed on these single-valued sets. In this paper we aim at linking these hyperstructures by constructing homomorphisms between them.

1. Motivation

Classical transformations such as Laplace, Carson-Laplace, Fourier and others are important mathematical tools with numerous useful applications. In [10] we applied the Laplace transform on integral Volterra operators with translation kernels in order to obtain a certain homomorphic mapping of the centralizer semihypergroup of such operators. In this paper, these ideas are going to be extended by some new results.

One of the basic properties of the Laplace transform apart from its linearity is the fact that it maps a convolution of original functions onto a product of their images. This enables us to construct the embedding of certain semihypergroups of Volterra integral operators with translation kernel (i.e. convolution integrals) into hypergroups of complex transformations.

Let us remind that

$$\xi\varphi(t) = f(t) + \lambda \int_a^t k(t,s)\varphi(s)ds \tag{1}$$

is called a Volterra equation (of the first kind if $\xi = 0$, and of the second kind if $\xi \neq 0$).

If the kernel $k(t,s)$ depends on the difference $t-s$ only, i.e. $k(t,s) = k(t-s)$ then the integral

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$$\int_0^t k(t,s)\varphi(s)ds = k(t) * \varphi(t)$$

is convolution of functions k and φ. If this is the case, the corresponding Volterra equation

$$\xi\varphi(t) = f(t) + \lambda \int_0^t k(t,s)\varphi(s)ds \tag{2}$$

transforms into

$$\xi\varphi(t) = f(t) + \lambda k(t) * \varphi(t),$$

which is in fact an equation of the convolution type. If the improper integral $\int_0^\infty e^{-pt}k(t) * \varphi(t)dt$ absolutely converges, then when applying the Laplace transform to convolution of functions k, φ, we obtain (with respect to the product theorem)

$$\mathcal{L}\{k(t) * \varphi(t)\} = \int_0^\infty e^{-pt} \int_0^t k(t,s)\varphi(s)dsdt = K(p)\Phi(p).$$

Consequently, after using the Laplace transform $\mathcal{L}\{f(t)\} = F(p)$, $\mathcal{L}\{\varphi(t)\} = \Phi(p)$ and $\mathcal{L}\{k(t)\} = K(p)$, the integral equation (2) transforms into the operational equation

$$\xi\Phi(p) = F(p) + \lambda K(p)\Phi(p) \tag{3}$$

This equation can be solved by usual classical methods. For details, a deeper explanation of the above concepts and an overall introduction in the issue of integral equations see e.g. [16].

Properties of the Laplace transformation and its application on Volterra integral operators were the motivation for the study of translation operators done in [7]. This connection can easily be expanded to include certain generalised affine transformation operators of continuous complex functions defined in the half plane of complex numbers.

Finally, study of some topics of physiscs, technology or astronomy often makes use of second-order differential equations of the Hill type, i.e. of equations of the form

$$y'' + [\Phi(x) + \lambda]y = 0 \tag{4}$$

with a periodic function $\Phi(x)$. In monograph [12], p. 411, there is included the Hill equation in the form

$$y'' + (a\mathrm{e}^{2x} + b\mathrm{e}^{x} + c)y = 0, \tag{5}$$

where the periodic function is $\Phi(x) = a\mathrm{e}^{2x} + b\mathrm{e}^{x}$, the period of which is complex and equals $2\pi i$. Further on in this paper we develop certain considerations concerning the Hill equation of this form which are included in [2].

In the paper we aim at providing a certain link between all the above mention concepts. Within this link we also include the linear homogeneous second- and third-order differential equations.

All the concepts and necessary definitions of basic notions of hyperstructure theory used in this paper may be found in [3, 4].

2. The hyperstructure construction tool

Hyperstructures constructed in this paper are on the EL type, i.e. hyperstructures constructed using the lemma known as the *Ends lemma*. Parts of the lemma we need for our considerations might be summed up as follows.

Lemma 2.1. [6, 15] *Let a triple $(G, \cdot, \leq)$ be a quasi-ordered semigroup. Define a hyperoperation*

$$\bullet : G \times G \to \mathcal{P}^*(G) \qquad \text{by} \qquad a \bullet b = [a \cdot b)_{\leq} = \{x \in G; a \cdot b \leq x\}$$

for all pairs of elements $a, b \in G$.

(1) *Then $(G, \bullet)$ is a semihypergroup which is commutative if the semigroup $(G, \cdot)$ is commutative.*
(2) *Let $(G, \bullet)$ be the above defined semihypergroup. Then $(G, \bullet)$ is a hypergroup if and only if for any pair of elements $a, b \in G$ there exists a pair of elements $c, c' \in G$ with a property $a \cdot c \leq b, c' \cdot a \leq b$.*

In [10], the following result concerning mappings of EL–hyperstructures was included as Lemma 2.

Lemma 2.2. *Let $(G, \cdot, \leq_G)$, $(H, \cdot, \leq_H)$ be quasi-ordered semigroups, $f : (G, \cdot, \leq_G) \to (H, \cdot, \leq_H)$ be an order-homomorphism, i.e. $f : (G, \cdot) \to (H, \cdot)$ is a homomorphism and simultaneously $f : (G, \leq_G) \to (H, \leq_H)$ is an isotone mapping. Furthermore, let $(G, \bullet_G)$ and $(H, \bullet_H)$ be EL–semihypergroups based on G, H respectively. Then $f : (G, \bullet_G) \to (H, \bullet_H)$ is an inclusion homomorphism.*

For more results on EL–hyperstructures cf. [13, 14, 15]. For uses of the construction cf. e.g. [1, 5, 7, 10, 11, 17]. For a deeper insight into the case of semigroups not being groups and yet creating hypergroups cf. [14].

3. EL–hyperstructures and their homomorphisms

3.1. **Volterra operators.** Consider now Volterra integral operators

$$V(\lambda, k, f)(\varphi) = \lambda \int_0^t k(t-s)\varphi(s)\mathrm{d}s + f(t), \tag{6}$$

where all continuous functions φ are of exponential order. By $\mathcal{V}_C(I)$ we denote the set of all such operators. Define the product of two such operators by

$$V(\lambda_1, k_1, f_1).V(\lambda_2, k_2, f_2) = V(\lambda_1\lambda_2, \lambda_1 k_2 + k_1, f_1 * f_2),$$

where

$$(f_1 * f_2)(x) = \int_0^x f_1(x-t)f_2(t)dt$$

is convolution of the given functions. Thus for $\varphi \in C(I)$ we have

$$V(\lambda_1, k_1, f_1).V(\lambda_2, k_2, f_2)(\varphi) = \lambda_1^2\lambda_2 \int_0^x k_2(x-t)\varphi(t)dt + f_1(x) * f_2(x) +$$
$$+\lambda_1\lambda_2 \int_0^x k_1(x-t)\varphi(t)dt = V(\lambda_1^2\lambda_2, k_2, f_1 * f_2)(\varphi) + V(\lambda_1\lambda_2, k_1, 0)(\varphi).$$

Evidently, such an operation is *not commutative*. It is not difficult to show that *it is associative*.

3.2. **Translation operators.** Suppose k, f, φ are continuous functions, where φ is of the bounded exponential growth on $I = \langle 0, \infty)$. After applying the Laplace transform

$$\mathcal{L}\{\varphi\} = \int_0^\infty e^{-pt}\varphi(t)dt,$$

on a Volterra integral operator $V(\lambda, k, f)(\varphi)$ we obtain

$$\mathcal{L}\{V(\lambda, k, f)(\varphi)\} = \lambda K(p)\Phi(p) + F(p).$$

Consider the half-plane of complex numbers $\Omega = \{z; Re z > 0\}$. For $\lambda \in \mathbb{R}^+$, $K, F \in \mathbb{C}(\Omega)$, $F(p)$ different from 0 for any $p \in \Omega$, we define $T(\lambda, K, F)\Phi(p) = \lambda K(p)\Phi(p) + F(p)$, $p \in \Omega$, $\Phi \in \mathbb{C}(\Omega)$. On the set $\mathcal{T}(\Omega)$ of such operators we consider the binary operation

$$T(\lambda, K, F).T(\mu, S, G)\Phi(p) = T(\lambda\mu, \lambda S + K, FG)\Phi(p) =$$
$$= \lambda^2\mu S(p)\Phi(p) + \lambda\mu K(p)\Phi(p) + F(p)G(p) =$$
$$= T(\lambda\mu, K, FG)\Phi(p) + T(\lambda^2\mu, S, 0)\Phi(p),$$

thus $T(\lambda, K, F).T(\mu, S, G) = T(\lambda\mu, \lambda S + K, FG)$.

This is a certain generalization of the concept of translation operators, which are investigated in paper [7]. In a similar way as in that paper it is easy to show in our case as well that our groupoid $(\mathcal{T}(\Omega), .)$ *is a noncommutative group*. Indeed, in the same way as in the case of $(\mathcal{V}_C(I), .)$, the above defined operation is associative; the operator $T(1, 0, 1)$, i.e. if $\lambda = 1$, $K(p) \equiv 0$, $F(p) \equiv 1$ is the unit, i.e. $T(1, 0, 1)\Phi(p) \equiv 1$ and $T(1, 0, 1).T(\lambda, K, F) = T(\lambda, K, F).T(1, 0, 1) = T(\lambda, K, F)$. For arbitrary $T(\lambda, K, F) \in \mathcal{T}(\Omega)$ its inverse operator is

$$T^{-1}(\lambda, K, F) = T(\frac{1}{\lambda}, -\frac{K}{\lambda}, \frac{1}{F}).$$

Now, with the use of a suitable ordering on $\mathcal{T}(\Omega)$ we – as has been shown in [7] – obtain the *transposition hypergroup* $(\mathcal{T}(\Omega), \bullet)$.

3.3. **Mapping of $\mathcal{V}_c(I)$ into $\mathcal{T}(\Omega)$.** Now define $\mathbf{L}(V(\lambda,k,f)) = T(\lambda,K,F)$ if $\mathcal{L}\{V(\varphi)\} = T(\Phi)$.

Theorem 3.1. *The Laplace transformation defined on the set of Volterra operators $\mathcal{V}_c(I)$ of the convolution type is an embedding (i.e. an injective homomorphism) $\mathbf{L}$ of the semigroup $(\mathcal{V}_c(I),.)$ into the group $(\mathcal{T}(\Omega),.)$.*

Proof. It is a well-known fact that the Laplace transformation restricted on the space of continuous functions with bounded exponential growth is injective. Now consider an arbitrary pair of operators

$$V(\lambda_1,k_1,f_1), V(\lambda_2,k_2,f_2) \in \mathcal{V}_c(I).$$

Then

$$\begin{aligned}\mathbf{L}(V(\lambda_1,k_1,f_1).V(\lambda_2,k_2,f_2))\mathcal{L}\{\varphi\} = \mathbf{L}(V((\lambda_1\lambda_2,\lambda_1k_2+k_1,f_2))\mathcal{L}\{\varphi\} =\\ = \mathcal{L}\{\lambda_1\lambda_2\textstyle\int_0^t(\lambda_1k_2(t-s))\varphi(s)ds) + f_1(t)*f_2(t)\} =\\ \lambda_1^2\lambda_2K_2(p)\Phi(p) + \lambda_1\lambda_2K_1(p)\Phi(p) + F_1(p)F_2(p) =\\ = T(\lambda_1^2\lambda_2,K_2,0)\Phi + T(\lambda_1\lambda_2,K_1,F_1F_2)\Phi = T(\lambda_1\lambda_2,\lambda_1K_2+K_1,F_1F_2)\Phi =\\ = (T(\lambda_1,K_1,F_1)T(\lambda_2,K_2,F_2))\Phi = \mathbf{L}(V(\lambda_1,k_1,f_1)).\mathbf{L}(V(\lambda_2,k_2,f_2))\mathcal{L}\{\varphi\}.\end{aligned}$$

Thus

$$\begin{aligned}\mathbf{L}(V(\lambda_1,k_1,f_1).V(\lambda_2,k_2,f_2)) = T(\lambda_1,K_1,F_1).T(\lambda_2,K_2,F_2) =\\ = \mathbf{L}(V(\lambda_1,k_1,f_1)).\mathbf{L}(V(\lambda_2,k_2,f_2)),\end{aligned}$$

where $\mathbf{L}(V(\lambda,k,f)) = T(\lambda,K,F)$. □

3.4. **Centralizer semihypergroup determined by $V(\lambda_0,k_0,f_0)$.** Now let us construct a centralizer semihypergroup defined by a given operator $V_0 = V(\lambda_0,k_0,f_0) \in \mathcal{V}_C$. Recall that for a suitable function φ we have

$$V(\lambda_0,k_0,f_0)(\varphi) = \lambda_0\int_0^t k_0(t-s)\varphi(s)ds + f_0(t) = \lambda_0k_0(t)*\varphi(t) + f_0(t)$$

Define

$$\begin{aligned}Ct_{\mathcal{V}}(V_0) = \{V(\lambda,k,f); V(\lambda,k,f) \in \mathcal{V}_C, V(\lambda,k,f).V_0 = V_0.V(\lambda,k,f)\} =\\ = \{V(\lambda,k,f); V(\lambda\lambda_0,\lambda k_0+k,f*f_0) = V(\lambda\lambda_0,\lambda_0k+k_0,f*f_0)\}\end{aligned}$$

It is to verify that for an arbitrary positive integer n the $n-$th iteration of the operator V_0 is $V_0^n = V(1,nk_0,f_0*\ldots*f_0)$ if $\lambda_0 = 1$ and $V_0^n = V(\lambda_0^n,\frac{\lambda_0^n-1}{\lambda_0-1}k_0,f_0*\ldots*f_0)$ if $\lambda_0 \neq 1$. The member $f_0*\ldots*f_0$ denotes the convolution of n fold f_0.

We define a binary hyperoperation $\bullet_{\mathcal{V}}$ on $Ct_{\mathcal{V}}(V_0)$ in the following way: Let $V_i = V(\lambda_i,k_i,f_i)$, $i = 0,1,2$, $V_i \in Ct_{\mathcal{V}}(V_0)$. We put

$$V_1 \bullet_{\mathcal{V}} V_2 = \{V_0^n\cdot V_1\cdot V_2,\ n\in\mathbb{N}_0\} = \{V_0^n\cdot V(\lambda_1\lambda_2,\lambda_1k_2+k_1,f_1*f_2),\ n\in\mathbb{N}_0\},$$

where $V_0^0(\varphi) = f_0$. Let us define a binary relation on the set $Ct_{\mathcal{V}}(V_0)$ such that $V_1 \leq V_2$ whenever there exists an integer $m \in \mathbb{N}_0$ with a property $V_2 = V_0^m\cdot V_1$. It is easy to see that such a relation is a quasi-ordering (i.e. is reflexive and transitive). Moreover, we get that $(Ct_{\mathcal{V}}(V_0),\cdot,\leq)$ *is a quasi-ordered semigroup.*

Indeed, suppose that V_1, V_2, $V \in Ct_{\mathcal{V}}(V_0)$ is such a triad that $V_1 \leq V_2$. Then for a suitable $m \in \mathbb{N}_0$ we have $V_2 = V_0^m \cdot V_1$. Further, $V \cdot V_2 = V \cdot V_0^m \cdot V_1 = V_0^m \cdot V \cdot V_1$. Therefore $V \cdot V_1 \leq V \cdot V_2$. In the same way we get that $V_1 \cdot V \leq V_2 \cdot V$, hence $(Ct_{\mathcal{V}}(V_0), \cdot, \leq)$ is a quasi-ordered semigroup. According to Lemma 2.1 we have that $(Ct_{\mathcal{V}}(V_0), \bullet_{\mathcal{V}})$ *is a semihypergroup.* It will be called a *centralizer semihypergroup determined by operator* V_0.

3.5. **Centralizer semihypergroup determined by** $T(\lambda_0, K_0, F_0)$**.** Now we construct a centralizer hypergroup $Ct_{\mathcal{T}}(T_0)$ of transformation operators $T(\lambda, K, F)$ determined by a given operator T_0. Notice that

$$T(\lambda, K, F)\Phi(p) = \lambda K(p)\Phi(p) + F(p),\ p \in \Omega.$$

For $T_0 = T(\lambda_0, K_0, F_0) \in \mathcal{T}(\Omega)$ we define

$$\begin{aligned} Ct_{\mathcal{T}}(T_0) &= \{T(\lambda, K, F);\ T(\lambda, K, F) \cdot T_0 = T_0 \cdot T(\lambda, K, F)\} = \\ &= \{T(\lambda, K, F);\ T(\lambda\lambda_0, \lambda K_0 + K, FF_0) = T(\lambda\lambda_0, \lambda_0 K + K_0, FF_0)\}. \end{aligned}$$

If $\lambda_0 \neq 1$ then $K(p) = \frac{\lambda-1}{\lambda_0-1}K_0(p),\ p \in \Omega$. If $\lambda_0 = 1$ then $\lambda = 1$. For an arbitrary $n \in \mathbb{N}$ we have

$$T^n(\lambda, K, F)\Phi = T^n(\lambda^n, \frac{\lambda^n - 1}{\lambda - 1}K, (F)^n)\Phi = \frac{\lambda^n(\lambda^n - 1)}{\lambda - 1}K(p)\Phi(p) + (F(p))^n$$

if $\lambda \neq 1$. If $\lambda = 1$ then

$$T^n(\lambda, K, F) = T(1, nK, (F)^n).$$

Further we define a binary hyperoperation $\bullet_{\mathcal{T}}$ on $Ct_{\mathcal{T}}(T_0)$ by

$$\begin{aligned} T(\lambda_1, K_1, F_1) \bullet_{\mathcal{T}} T(\lambda_2, K_2, F_2) &= \{T_0^n \cdot T(\lambda_1, K_1, F_1) \cdot T(\lambda_2, K_2, F_2),\ n \in \mathbb{N}_0\} = \\ &= \{T_0^n \cdot T(\lambda_1\lambda_2, \lambda_1 K_2 + K_1, F_1F_2),\ n \in \mathbb{N}_0\}. \end{aligned}$$

We could show in a similar way as for $(Ct_{\mathcal{V}}(V_0), \bullet_{\mathcal{V}})$ (again using Lemma 2.1) that $(Ct_{\mathcal{T}}(T_0), \bullet_{\mathcal{T}})$ *is a noncommutative hypergroup.*

3.6. **Homomorphisms of centralizer hyperstructures.**

Theorem 3.2. *Suppose* λ_0 *is a (complex) number,* $\lambda_0 \neq 0$, $k_0, f_0 \in \mathbb{C}(\langle 0, \infty))$ *are originals of the Laplace transform such that the integral* $\int_0^\infty e^{-pt}k_0(t) * \varphi(t)dt$ *is absolutely converging,* $\varphi \in \mathcal{F}_{k_0} \subset \mathbb{C}(\langle 0, \infty))$, $V_0 = V(\lambda_0, k_0, f_0)$*. If for any* $V(\lambda, k, f) \in Ct_{\mathcal{V}}(V_0)$*, where* λ, k, f *satisfy the above conditions, we define*

$$\mathbf{L}(V(\lambda, k, f)) = T(\lambda, K, F),$$

where

$$\mathcal{L}\{V(\lambda, k, f)(\varphi)\} = T(\lambda, \mathcal{L}\{k\}, \mathcal{L}\{f\})\mathcal{L}\{\varphi\} = T(\lambda, K, F)\mathcal{L}\{\varphi\},$$

then

$$\mathbf{L} : Ct_{\mathcal{V}}(V_0) \to Ct_{\mathcal{T}}(T_0) \qquad (T_0 = \mathbf{L}(V_0))$$

is an inclusion embedding (i.e. an inclusive injective homomorphism) of the semihypergroup $(Ct_{\mathcal{V}}(V_0), \bullet_{\mathcal{V}})$ *into the hypergroup* $(Ct_{\mathcal{T}}(T_0), \bullet_{\mathcal{T}})$*.*

Proof. According to Theorem 3.1 the mapping $\mathbf{L} : (Ct_{\mathcal{V}}(V_0)) \to (Ct_{\mathcal{T}}(T_0))$ is an injective homomorphism of the semihypergroup $(Ct_{\mathcal{V}}(V_0), \cdot)$ into the group $(Ct_{\mathcal{T}}(T_0), \cdot)$ for suitable fixed operators $V_0 \in \mathcal{V}_C(I)$, $T_0 \in \mathcal{T}(\Omega)$. We have to verify that $\mathbf{L} : (Ct_{\mathcal{V}}(V_0), \leq_{\mathcal{V}}) \to (Ct_{\mathcal{T}}(T_0), \leq_{\mathcal{T}})$ is an isotone mapping. Recall that for $V_1, V_2 \in \mathcal{V}_C(I)$ we put $V_1 \leq_{\mathcal{V}} V_2$ whenever $V_2 = V_0^m \cdot V_1$ for some integer $m \in \mathbb{N}_0$ and similarly $T_1 \leq_{\mathcal{T}} T_2$ whenever there is $n \in \mathbb{N}_0$ with the property $T_2 = T_0^n \cdot T_1$. Therefore, if $V_1 \leq_{\mathcal{V}} V_2$, then by Theorem 3.1

$$T_2 = \mathbf{L}(V_2) = \mathbf{L}(V_0^m \cdot V_1) = \mathbf{L}(V_0)^m \cdot \mathbf{L}(V_1) = [\mathbf{L}(V_0)]^m \cdot \mathbf{L}(V_1) = T_0^m \cdot T_1,$$

where $\mathbf{L}(V_i) = T_i,\ i = 0, 1, 2$. Hence $T_1 \leq_{\mathcal{T}} T_2$. By Lemma 2.2 we obtain that $\mathbf{L} : (Ct_{\mathcal{V}}(V_0), \bullet_{\mathcal{V}}) \to (Ct_{\mathcal{T}}(T_0), \bullet_{\mathcal{T}})$ is an inclusion embedding of the semihypergroup $(Ct_{\mathcal{V}}(V_0)\bullet_{\mathcal{T}})$ into the hypergroup $(Ct_{\mathcal{T}}(T_0), \bullet_{\mathcal{T}})$. □

3.7. **Hyperstructures of linear homogeneous third-order ODE's.** Consider differential operators formed by left-hand sides of linear homogeneous third-order differential equations of the form

$$y'''(x) + \sum_{k=0}^{2} p_k(x) y^{(k)}(x) = 0,$$

where coefficients $p_k \in C(I)$ for some open interval $I \subseteq \mathbb{R}$. Denote by $L(p_0(x), p_1(x), p_2(x))$ the linear differential operator such that

$$L(p_0(x), p_1(x), p_2(x)) y(x) = y'''(x) + \sum_{k=0}^{2} p_k(x) y^{(k)}(x)$$

for any function $y \in C^3(I)$. Denote by $\mathbb{LA}_3(I)$ the set of all such operators (with $p_0(x) > 0, x \in I$) endowed with the binary operation defined by

$$L(p_0, p_1, p_2) \cdot L(q_0, q_1, q_2) = L(p_0 q_0, p_0 q_1 + p_1, p_0 q_2 + q_1),$$

where the products of $p_i, q_i,\ i = 0, 1, 2$, are pointwise products of functions for all $x \in I$.

Let $L(p_0, p_1, p_2),\ L(q_0, q_1, q_2) \in \mathbb{LA}_3(I)$ be an arbitrary pair of operators. We define $L(p_0, p_1, p_2) \leq L(q_0, q_1, q_2)$ if $p_0(x) = q_0(x)$, $p_1(x) \leq q_1(x)$, $p_2(x) \leq q_2(x)$ for all $x \in I$. Similarly as in [9] we obtain the following theorem.

Theorem 3.3. *Let $I \subseteq \mathbb{R}$ be an open interval. Then $(\mathbb{LA}_3(I), \cdot, \leq)$ is a noncommutative partially ordered group with the neutral element $L(1, 0, 0)$ (here $L(1, 0, 0) y(x) = y'''(x) + y(x)$ for any function $y \in C^3(I)$). The inverse element to $L(p_0(x), p_1(x), p_2(x))$ is*

$$L^{-1}(p_0(x), p_1(x), p_2(x)) = L(\frac{1}{p_0(x)}, -\frac{p_1(x)}{p_0(x)}, -\frac{p_2(x)}{p_0(x)}), x \in I.$$

Proof. Evidently $L(1, 0, 0) \cdot L(p_0, p_1, p_2) = L(p_0, p_1, p_2) \cdot L(1, 0, 0)$. Further,

$$L(\frac{1}{p_0}, -\frac{p_1}{p_0}, -\frac{p_2}{p_0}) \cdot L(p_0, p_1, p_2) = L(1, 0, 0) = L(p_0, p_1, p_2) \cdot L(\frac{1}{p_0}, -\frac{p_1}{p_0}, -\frac{p_2}{p_0}),$$

thus $L^{-1}(p_0, p_1, p_2) = L(\frac{1}{p_0}, -\frac{p_1}{p_0}, -\frac{p_2}{p_0})$.

Now, for $L(p_0, p_1, p_2), L(q_0, q_1, q_2) \in \mathbb{LA}_3(I)$ we have

$$L(p_0, p_1, p_2) \cdot L^{-1}(q_0, q, q_2) = L(p_0, p_1, p_2) \cdot L(\frac{1}{q_0}, -\frac{q_1}{q_0}, -\frac{q_2}{q_0}) \in \mathbb{LA}_3(I).$$

Considering the above defined ordering we obtain easily that $(\mathbb{LA}_3(I), \cdot, \leq)$ is a partially ordered group. □

Now denote

$$\mathbb{L}_C\mathbb{A}_3(I) = \{L(r, p_1, p_2); p_1, p_2 \in C(I), r \in \mathbb{R}, r > 0\}$$

and

$$\mathbb{L}_1\mathbb{A}_3(I) = \{L(1, p_1, p_2); p_1, p_2 \in C(I)\}.$$

For an arbitrary pair of operators $L(r, p_1, p_2)$, $L(s, q_1, q_2) \in \mathbb{L}_C\mathbb{A}_3(I)$ we have

$$L(r, p_1, p_2) \cdot L^{-1}(s, q_1, q_2) = L(\frac{r}{s}, \frac{p_1 - rq_1}{s}, \frac{p_2 - rq_2}{s}) \in \mathbb{L}_C\mathbb{A}_3(I).$$

Moreover, $L(1, 0, 0) \in \mathbb{L}_C\mathbb{A}_3(I)$, thus $(\mathbb{L}_C\mathbb{A}_3(I), \cdot)$ *is a subgroup of the group* $(\mathbb{LA}_3(I), \cdot)$ and similarly $(\mathbb{L}_1\mathbb{A}_3(I), \cdot)$ *is a subgroup of the group* $(\mathbb{L}_C\mathbb{A}_3(I), \cdot)$. Furthermore, by a detailed calculation we obtain the following theorem. Notice that the symbol $A \triangleleft B$ reads "A is a normal, i.e. invariant, subgroup of B".

Theorem 3.4. *Let $I \subseteq \mathbb{R}$ be an open interval. Groups $(\mathbb{L}_1\mathbb{A}_3(I), \cdot)$, $(\mathbb{L}_C\mathbb{A}_3(I), \cdot)$ are subgroups of the group $(\mathbb{LA}_3(I), \cdot)$. Moreover, $(\mathbb{L}_1\mathbb{A}_3(I), \cdot)$ is commutative. We also have*

$$(\mathbb{L}_1\mathbb{A}_3(I), \cdot) \triangleleft (\mathbb{L}_C\mathbb{A}_3(I), \cdot) \triangleleft (\mathbb{LA}_3(I), \cdot)$$
$$(\mathbb{L}_1\mathbb{A}_3(I), \cdot) \triangleleft (\mathbb{LA}_3(I), \cdot) \qquad .$$

Proof. For arbitrary operators $L(r, p_1, p_2)$, $L(s, q_1, q_2) \in \mathbb{L}_C\mathbb{A}_3(I)$ we have

$$L(r, p_1, p_2) \cdot L^{-1}(s, q_1, q_2) = L(\frac{r}{s}, \frac{p_1 - rq_1}{s}, \frac{p_2 - rq_2}{s}) \in \mathbb{L}_C\mathbb{A}_3(I),$$

$L(1, 0, 0) \in \mathbb{L}_C\mathbb{A}_3(I)$, thus $(\mathbb{L}_C\mathbb{A}_3(I), \cdot)$ is a subgroup of the group $(\mathbb{LA}_3(I), \cdot)$. Similarly we obtain that $(\mathbb{L}_1\mathbb{A}_3(I), \cdot)$ is a subgroup of the group $(\mathbb{L}_C\mathbb{A}_3(I), \cdot)$.

If $L(1, p_1, p_2), L(1, q_1, q_2) \in \mathbb{L}_1\mathbb{A}_3(I)$, then

$$L(1, p_1, p_2) \cdot L(1, q_1, q_2) = L(1, q_1 + p_1, q_2 + p_2) = L(1, q_1, q_2) \cdot L(1, p_1, p_2),$$

i.e. the group $(\mathbb{L}_1\mathbb{A}_3(I), \cdot)$ is commutative.

Now, suppose $L(r, q_1, q_2) \in \mathbb{L}_C\mathbb{A}_3(I)$, $L(p_0, p_1, p_2) \in \mathbb{LA}_3(I)$ are arbitrary operators. We have

$$L^{-1}(p_0, p_1, p_2) \cdot L(r, q_1, q_2) \cdot L(p_0, p_1, p_2) = L(\tfrac{1}{p_0}, -\tfrac{p_1}{p_0}, -\tfrac{p_2}{p_0}) \cdot L(rp_0, rp_1 + q_1, rp_2 + q_2) =$$
$$= L(r, \tfrac{(r-1)p_1 + q_1}{p_0}, \tfrac{(r-1)p_2 + q_2}{p_0}) \in \mathbb{L}_C\mathbb{A}_3(I),$$

thus $L^{-1}(p_0, p_1, p_2) \cdot \mathbb{L}_C\mathbb{A}_3(I) \cdot L(p_0, p_1, p_2) \subseteq \mathbb{L}_C\mathbb{A}_3(I)$, hence the group $(\mathbb{L}_C\mathbb{A}_3(I), \cdot)$ is an invariant subgroup of the group $(\mathbb{LA}_3(I), \cdot)$. Similarly,

$$L^{-1}(p_0, p_1, p_2) \cdot L(1, q_1, q_2) \cdot L(p_0, p_1, p_2) = L(\tfrac{1}{p_0}, -\tfrac{p_1}{p_0}, -\tfrac{p_2}{p_0}) \cdot L(p_0, p_1 + q_1, p_2 + q_2) =$$
$$L(1, \tfrac{q_1}{p_0}, \tfrac{q_2}{p_0}) \in \mathbb{L}_1\mathbb{A}_3(I),$$

which means that $(\mathbb{L}_1\mathbb{A}_3(I), \cdot) \triangleleft (\mathbb{L}\mathbb{A}_3(I), \cdot)$.

Finally, for an arbitrary operator $L(r, p_1, p_2) \in \mathbb{L}_C\mathbb{A}_3(I)$ we have

$$L^{-1}(r, p_1, p_2) \cdot L(1, q_1, q_2) \cdot L(r, p_1, p_2) = L(\tfrac{1}{r}, -\tfrac{p_1}{r}, -\tfrac{p_2}{r}) \cdot L(r, p_1 + q_1, p_2 + q_2) = \\ = L(1, \tfrac{q_1}{r}, \tfrac{q_2}{r}) \in \mathbb{L}_1\mathbb{A}_3(I),$$

which implies $L^{-1}(r, p_1, p_2) \cdot \mathbb{L}_1\mathbb{A}_3(I) \cdot L(r, p_1, p_2) \subseteq \mathbb{L}_1\mathbb{A}_3(I)$, i.e. $(\mathbb{L}_1\mathbb{A}_3(I), \cdot) \triangleleft (\mathbb{L}_C\mathbb{A}_3(I), \cdot)$. □

3.8. **A mapping from $\mathbb{L}\mathbb{A}_3(I)$ to $\mathbb{L}\mathbb{A}_2(I)$.** In the previous subsection we have studied operators formed by left-hand sides of linear homogeneous third-order differential operators. In [8] similar reasoning was performed for *second-order* equations. There, the set of operators was defined as

$$\mathbb{L}\mathbb{A}_2(I) = \{L(p_0, p_1); p_0, p_1 \in C(I), p_0(x) > 0 \text{ for all } x \in I\}$$

and the multiplication of operators as

$$L(p_0, p_1) \cdot L(q_0, q_1) = L(p_0q_0, p_0q_1 + p_1).$$

It was also proved that $(\mathbb{L}\mathbb{A}_2(I), \cdot)$ *is a noncommutative group*. For more results concerning $\mathbb{L}\mathbb{A}_2(I)$ cf. [8].

Define now a mapping $F : \mathbb{L}\mathbb{A}_3(I) \to \mathbb{L}\mathbb{A}_2(I)$ by

$$F(L(p_0, p_1, p_2)) = L(p_0, p_1) \in \mathbb{L}\mathbb{A}_2(I)$$

for an arbitrary operator $L(p_0, p_1, p_2) \in \mathbb{L}\mathbb{A}_3(I)$. If $L(p_0, p_1, p_2)$, $L(q_0, q_1, q_2) \in \mathbb{L}\mathbb{A}_3(I)$, we have

$$F(L(p_0, p_1, p_2) \cdot L(q_0, q_1, q_2)) = F(L(p_0q_0, p_0q_1 + p_1, p_0q_2 + p_2)) = \\ L(p_0q_0, p_0q_1 + p_1) = L(p_0, p_1) \cdot L(q_0, q_1) = F(L(p_0, p_1, p_2)) \cdot F(L(q_0, q_1, q_2)).$$

Moreover, $F(L(1, 0, 0)) = L(1, 0)$. If $L(p_0, p_1, p_2) \leq L(q_0, q_1, q_2)$, i.e. $p_0(x) = q_0(x)$, $p_1(x) \leq q_1(x)$, $p_2(x) \leq q_2(x)$, $x \in I$, we have

$$L(p_0, p_1) \leq L(q_0, q_1)$$

as operators in the partially ordered group $(\mathbb{L}\mathbb{A}_2(I), \cdot)$, thus the mapping $F : \mathbb{L}\mathbb{A}_3(I) \to \mathbb{L}\mathbb{A}_2(I)$ is isotone (order-preserving). Hence we get the following lemma:

Lemma 3.5. *Mapping $F : (\mathbb{L}\mathbb{A}_3(I), \cdot, \leq) \to (\mathbb{L}\mathbb{A}_2(I), \cdot, \leq)$ is an order-homomorphism of the partially ordered group $(\mathbb{L}\mathbb{A}_3(I), \cdot, \leq)$ onto the partially ordered group $(\mathbb{L}\mathbb{A}_2(I), \cdot, \leq)$, the restriction on $(\mathbb{L}_1\mathbb{A}_3(I), \cdot)$ is an order-preserving embedding (i.e. is injective).*

Similarly as in [8] we define binary hyperoperations on the considered groups by the *Ends lemma*. For any pair of operators $L(p_0, p_1, p_2) \leq L(q_0, q_1, q_2) \in \mathbb{L}\mathbb{A}_3(I)$ we put

$$L(p_0, p_1, p_2) * L(q_0, q_1, q_2) = \\ = \{L(u, v, w) \in \mathbb{L}\mathbb{A}_3(I); L(p_0, p_1, p_2) \cdot L(q_0, q_1, q_2) \leq L(u, v, w)\} = \\ = \{L(p_0q_0, v, w); v, w \in C(I), p_0q_1 + p_1 \leq v, p_0q_2 + p_2 \leq w\}.$$

By direct calculation or by [17], p. 268, Theorem 4 and Lemma 2.2, we obtain the following result.

Theorem 3.6. *Let $I \subseteq \mathbb{R}$ be an open interval. The mapping F is an inclusion homomorphism of the transposition hypergroup $(\mathbb{LA}_3(I), *)$ into the transposition hypergroup $(\mathbb{LA}_2(I), *)$. Its restriction onto the subhypergroup $(\mathbb{L}_1\mathbb{A}_3(I), *)$ of the hypergroup $(\mathbb{LA}_3(I), *)$ is injective.*

3.9. **Generalized affine transformations.** Now consider the set of generalized affine transformations $T(\lambda, K, F)$, where functions K, F are restricted onto some real interval $I \subseteq \mathbb{R}$ within a complex domain Ω. Defining a binary operation on the set $\mathcal{T}(I) = \{T(\lambda, K, F); K \uparrow I, F \uparrow I\}$ by

$$T(\lambda_1, K_1, F_1) \cdot T(\lambda_2, K_2, F_2) = T(\lambda_1\lambda_2, \lambda_1 K_2 + K_1, \lambda_1 F_2 + F_1)$$

we obtain (similarly as in [7]) that the groupoid $(\mathcal{T}(I), \cdot)$ is a noncommutative group.

Denote by $\Phi : \mathcal{T}(I) \to \mathbb{LA}_3(I)$ the mapping defined by

$$\Phi(T(\lambda, K, F)) = L(\lambda, K, F) \in \mathbb{LA}_3(I) \tag{7}$$

for any operator $T(\lambda, K, F) \in \mathcal{T}(I)$. In other words, for any function $y \in C^3(I)$ there holds

$$L(\lambda, K, F)y(x) = y'''(x) + F(x)y''(x) + K(x)y'(x) + \lambda y(x).$$

Then Φ is an isomorphism of the group $(\mathcal{T}(I), \cdot)$ onto the group $(\mathbb{L}_C\mathbb{A}_3(I), \cdot)$. In a similar fashion as above we can construct the hypergroupoid $(\mathcal{T}(I), *)$. In detail, if $T(\lambda_1, K_1, F_1)$, $T(\lambda_2, K_2, F_2) \in \mathcal{T}(I)$, we put $T(\lambda_1, K_1, F_1) \leq T(\lambda_2, K_2, F_2)$ whenever $\lambda_1 = \lambda_2$, $K_1(x) \leq K_2(x)$, $F_1(x) \leq F_2(x)$, $x \in I$. Then for any pair of translations $T(\lambda_1, K_1, F_1)$, $T(\lambda_2, K_2, F_2) \in \mathcal{T}(I)$ we define a binary hyperoperation

$$T(\lambda_1, K_1, F_1) * T(\lambda_2, K_2, F_2) = \{T(\lambda, K, F); T(\lambda_1, K_1, F_1) \cdot T(\lambda_2, K_2, F_2) \leq T(\lambda, K, F)\}$$

and obtain (again using Lemma 2.1) that the hypergroupoid $(\mathcal{T}(I), *)$ *is a noncommutative transposition hypergroup.* Consequently, we obtain the following theorem.

Theorem 3.7. *The mapping $\Phi : \mathcal{T}(I) \to \mathbb{LA}_3(I)$ is an injective inclusion homomorphism of the transposition hypergroup $(\mathcal{T}(I), *)$ into the transposition hypergroup $(\mathbb{LA}_3(I), *)$.*

Remark 3.8. The above presented considerations can be in a very simple way extended onto rings of complex functions with real values, i.e. functions $\varphi : \Omega \to \mathbb{R}$, where $\Omega \subseteq \mathbb{C}$.

3.10. **Second-order Hill type ODE.** Recall now the Hill equation in the form (4), i.e.

$$y'' + [\Phi(x) + \lambda]y = 0.$$

For any triad of real or complex numbers a, b, c we define

$$L(0, (a, b, c))y = y'' + (a\mathrm{e}^{2x} + b\mathrm{e}^{x} + c)y, \tag{8}$$

$y \in C^2(I)$ and put

$$\mathbb{H}_2(I) = \{L(0, (a, b, c)); c \in \mathbb{C}, c \neq 0\}. \tag{9}$$

Various binary operations on the set $\mathbb{H}_2(I)$ may be defined. Suppose that $L(0,(a_0,a_1,a_2))$, $L(0,(b_0,b_1,b_2)) \in \mathbb{H}_2(I)$ with $a_2 \neq 0 \neq b_2$. Define

$$L(0,(a_0,a_1,a_2)) \cdot L(0,(b_0,b_1,b_2)) = L(0,(a_2b_0+a_0, a_2b_1+a_1, a_2b_2)). \tag{10}$$

It can be easily verified that the above binary operation creates on $\mathbb{H}_2(I)$ the structure of a *non-commutative group* with the unit $L(0,(0,0,1))$. For every $L(0,(a,b,c)) \in \mathbb{H}_2(I)$ we have the inverse

$$L^{-1}(0,(a,b,c)) = L(0,(-\frac{a}{c}, -\frac{b}{c}, \frac{1}{c})) \in \mathbb{H}_2(I).$$

Now consider operators $L(0,(a,b,c)) \in \mathbb{H}_2(I)$ with all coefficients real, $c > 0$ and apply the *Ends lemma*, i.e. Lemma 2.1, on the partially ordered group $(\mathbb{H}_2(I), \cdot, \leq)$, where for $L(0,(a_1,b_1,c_1))$, $L(0,(a_2,b_2,c_2)) \in \mathbb{H}_2(I)$ we define

$$L(0,(a_1,b_1,c_1)) \leq L(0,(a_2,b_2,c_2)) \text{ if } c_1 = c_2, a_1 \leq a_2, b_1 \leq b_2. \tag{11}$$

Then we immediately obtain that $(\mathbb{H}_2(I), *)$ is a *non-commutative transposition hypergroup.* Moreover, define a mapping $\Psi : \mathbb{H}_2(I) \to \mathcal{T}(I)$ by

$$\Psi(L(0,(a,b,c))) = T(c,a,b) \tag{12}$$

for any operator $L(0,(a,b,c)) \in \mathbb{H}_2(I)$. Then we have the following result.

Theorem 3.9. *Let $(\mathcal{T}(I), *)$, $(\mathbb{H}_2(I), *)$ be transposition hypergroups defined above. Then the mapping $\Psi : \mathbb{H}_2(I) \to \mathcal{T}(I)$ is an isomorphic embedding of the hypergroup $(\mathbb{H}_2, *)$ onto the subhypergroup $\Psi(\mathbb{H}_2(I))$ of the hypergroup $(\mathcal{T}(I), *)$ of all operators $T(\lambda, K, F)$ with constant functions K, F.*

Proof. The mapping $\Psi : \mathbb{H}_2(I) \to \mathcal{T}(I)$ is evidently injective. Further, if $L(0,(a_1,b_1,c_1))$, $L(0,(a_2,b_2,c_2)) \in \mathbb{H}_2(I)$, then

$$\begin{aligned}\Psi(L(0,(a_1,b_1,c_1)) \cdot L(0,(a_2,b_2,c_2))) = \Psi(L(0,(c_1a_2+a_1, c_1b_2+b_1, c_1c_2))) = \\ = T(c_1c_2, c_1a_2+a_1, c_1b_2+b_1) = T(c_1,a_1,b_1) \cdot T(c_2,a_2,b_2) = \\ = \Psi(L(0,a_1,b_1,c_1)) \cdot \Psi(L(0,(a_2,b_2,c_2))),\end{aligned}$$

i.e. the mapping Ψ is a group embedding. Since Ψ is also order-preserving, we obtain that this mapping is an isomorphic embedding of the hypergroup $(\mathbb{H}_2(I), *)$ into the hypergroup $(\mathcal{T}(I), *)$. □

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