GENERALIZED k, m-STEP FIBONACCI SEQUENCES AND MATRICES

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Abstract. For two given integers k, m and the nonnegative real constants c_i, $i = 1, 2, \dots, k$, we introduce the *generalized k, m-step Fibonacci sequence* by presenting a recursive formula that generates the n-th term as the sum of k successive previous terms starting the sum at the m-th previous term. The *generalized k, m-Fibonacci matrices* are defined, their characteristic polynomials are presented and some bounds for the spectral radius and the modulus of the remaining eigenvalues of the matrices are discussed. A closed formula of the n-th term of generalized k, m-step Fibonacci sequence is given generalizing the Binet' s formula. Two limiting properties of the sequence are presented. The limits are related to the spectral radius of the generalized k, m-Fibonacci matrices.

1. Introduction

Fibonacci numbers are one of the best-known numerical sequences and have many important applications to a wide variety of research areas such as mathematics, computer science, physics, biology, and statistics. For the applications and the theory of Fibonacci numbers see, e.g. [1, 2, 3, 5, 7, 6, 8] and the references given therein. In [1, 2, 8], the well-known Fibonacci sequence is formulated by the recurrence relation $f_n = f_{n-1} + f_{n-2}$, $n \geq 3$, with $f_1 = f_2 = 1$.
Many authors have considered and discussed the extending/generalizing of the above definition as in the following:

- the k-step Fibonacci sequence is derived by the recurrence relation, $f_n = f_{n-1} + f_{n-2} + \cdots + f_{n-k}$, $n \geq k+1$, with $f_1 = f_2 = \ldots = f_k = 1$, [1, 7, 8],

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- the generalized k-step Fibonacci sequence is derived by the recurrence relation, $f_n = c_1 f_{n-1} + c_2 f_{n-2} + \cdots + c_k f_{n-k}$, $n \geq k+1$, with $f_1 = f_2 = \ldots = f_k = 1$, and $c_1, c_2, \ldots, c_k$ are arbitrary real numbers, [2, 3, 5, 6, 10].

In the present paper, we define the generalized k,m-step Fibonacci sequences and the associated generalized k,m-Fibonacci matrices and a closed formula for the generalized k,m-step Fibonacci sequences is derived as well. Two limiting properties concerning the generalized k,m-step Fibonacci sequences are obtained, related to the spectral radius of the generalized k,m-Fibonacci matrices.

2. Generalized k,m-step Fibonacci sequences and Fibonacci matrices

For the integers $k = 1, 2, \ldots$, $m = 0, 1, \ldots$, and the real numbers $c_1, c_2, \ldots, c_k$, where c_1 is a positive constant and $c_2, c_3, \ldots, c_k$ are nonnegative real constants, we define the *generalized k,m-step Fibonacci sequence*, $\left(f_n^{\{k,m\}}(c_1, c_2, \ldots, c_k)\right)_{n=1,2,\ldots}$, whose n-th term f_n is given by the recursive formulation

$$\begin{aligned} f_n &= c_1 f_{n-m-1} + c_2 f_{n-m-2} + \cdots + c_k f_{n-m-k} \\ &= \sum_{j=m+1}^{k+m} c_{j-m} f_{n-j}, \quad \text{for every} \quad n \geq k+m+1, \end{aligned} \tag{1}$$

with

$$f_1 = f_2 = \ldots = f_{k+m} = 1. \tag{2}$$

From $c_1 > 0$, $c_2, c_3, \ldots, c_k \geq 0$ and (1)-(2), it is obvious that all the terms f_n of the generalized k,m-step Fibonacci sequence $\left(f_n^{\{k,m\}}(c_1, \ldots, c_k)\right)_{n=1,2,\ldots}$ are positive numbers and f_n is the sum of k terms starting the sum at the m-th previous term from f_n; thus equation (1) can be written equivalently as,

$$\begin{aligned} f_n &= c_1 f_{n-m-1} + c_2 f_{n-m-2} + \cdots + c_k f_{n-m-k} \\ &= \sum_{i=1}^{k} c_i f_{n-m-i}, \quad \text{for every} \quad n \geq k+m+1. \end{aligned} \tag{3}$$

For $m = 0$, the n-th term f_n of the *generalized k-step Fibonacci sequence* $\left(f_n^{\{k,0\}}(c_1, c_2, \ldots, c_k)\right)_{n=1,2,\ldots}$ is given by

$$\begin{aligned} f_n &= c_1 f_{n-1} + c_2 f_{n-2} + \cdots + c_{k-1} f_{n-k+1} + c_k f_{n-k} \\ &= \sum_{i=1}^{k} c_i f_{n-i}, \quad \text{for every} \quad n \geq k+1, \end{aligned} \tag{4}$$

with initial values

$$f_1 = f_2 = \ldots = f_k = 1. \tag{5}$$

Remark 2.1. **(i):** From (2)-(3) it is evident that for $k=1$ and $m=0,1,\ldots$, the n-th term f_n of the associated generalized Fibonacci sequence

$$\left(f_n^{\{1,m\}}(c_1)\right)_{n=1,2,\ldots}$$

is equal to $f_n = c_1^n$. Hereafter consider $k \geq 2$, since the case $k=1$ is trivial.

(ii): For $c_1 = c_2 = \ldots = c_k = 1$ the generalized k,m-step Fibonacci sequence $\left(f_n^{\{k,m\}}(1,1,\ldots,1)\right)_{n=1,2,\ldots}$ gives known sequences for various values of the steps k,m:
-for $k=2$, $m=0$, the equations (4)-(5) give the well-known Fibonacci sequence, $1,1,2,3,5,8,13,\ldots$, [1, 8].
-for $k=2$, $m=1$, the equations (2)-(3) give the Padovan sequence, $1,1,1,2,2,3,4,5,7,9,\ldots$, [1].

(iii): Notice that the terms of the sequence $\left(f_n^{\{k,m\}}(0,c_2,\ldots,c_k)\right)_{n=1,2,\ldots}$ are identified with the terms of the sequence $\left(f_n^{\{k-1,m+1\}}(c_2,\ldots,c_k)\right)_{n=1,2,\ldots}$, an observation that led us to formulate the restriction $c_1 \neq 0$ in the definition in (1) and its equivalent formula in (3). Moreover, notice that for $c_1 > 0$, and $c_2 = c_3 = \ldots = c_k = 0$, the terms of the sequence $\left(f_n^{\{k,m\}}(c_1,0,\ldots,0)\right)_{n=1,2,\ldots}$ have the same pattern. Hereafter consider at least two nonzero coefficients c_i, $i=1,2,\ldots,k$, in (3) because otherwise we have a trivial case.

Consider $k \geq 2$, $m=0$, $c_1 > 0$, $c_2,c_3,\ldots,c_k \geq 0$, and using (4), the generalized k-step Fibonacci sequence $\left(f_n^{\{k,0\}}(c_1,c_2,\ldots,c_k)\right)_{n=1,2,\ldots}$ can be represented by a $k\times k$ matrix, $Q_k(c_1,c_2,\ldots,c_k)$, which is defined as

$$Q_k(c_1,c_2,\ldots,c_k) = \begin{bmatrix} c_1 & c_2 & c_3 & \cdots & c_k \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix} \tag{6}$$

where the $1\times k$ first row has entries the real numbers $c_1,\ldots,c_k$. In the following, $Q_k(c_1,c_2,\ldots,c_k)$ is called *generalized k-Fibonacci matrix.*

Consider $k \geq 2$, $m \geq 1$, $c_1 > 0$, $c_2,c_3,\ldots,c_k \geq 0$, and using (3), the generalized k,m-step Fibonacci sequence $\left(f_n^{\{k,m\}}(c_1,\ldots,c_k)\right)_{n=1,2,\ldots}$ can be represented by a

$(k+m) \times (k+m)$ matrix, $R_{k,m}(c_1, c_2, \ldots, c_k)$, which is a block matrix such that

$$R_{k,m}(c_1, c_2, \ldots, c_k) = \begin{bmatrix} R_1 & R_2 \\ R_3 & R_4 \end{bmatrix}$$

$$(7) \qquad = \begin{bmatrix} 0 & 0 & \cdots & 0 & c_1 & \cdots & c_{k-1} & c_k \\ 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & \cdots & & 0 \\ 0 & 0 & 0 & 1 & 0 & \cdots & & \vdots \\ \vdots & & & & \ddots & \ddots & & \vdots \\ \vdots & & & & & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & \cdots & & 0 & 1 & 0 \end{bmatrix}$$

where the first row consists of the vector-matrices R_1, R_2; the m entries of the $1 \times m$ vector R_1 are equal to zero and the rest k entries of the $1 \times k$ vector R_2 are the real numbers $c_1, c_2, \ldots, c_k$; the $(k+m-1) \times (k+m-1)$ matrix R_3 is the identity matrix and the $(k+m-1) \times 1$ vector R_4 is equal to zero vector. In the following, $R_{k,m}(c_1, c_2, \ldots, c_k)$ is called *generalized k, m-Fibonacci matrix.*

Remark 2.2. **(i):** The generalized k-Fibonacci matrix $Q_k(c_1, c_2, \ldots, c_k)$ in (6) has been presented in [5] and its determinant has been computed in [3, 6].

(ii): For $c_1 = c_2 = \ldots = c_k = 1$, the associated generalized k-Fibonacci matrix $Q_k(1, 1, \ldots, 1)$ in (6) has been discussed in [1, 7] and the associated generalized k, m-Fibonacci matrix $R_{k,m}(1, 1, \ldots, 1)$ in (7) has been investigated in [1].

(iii): The trace of a matrix A is denoted by $tr(A)$. It is evident from (6) and (7), respectively, that

$$tr(Q_k(c_1, c_2, \ldots, c_k)) = c_1, \quad \text{and} \quad tr(R_{k,m}(c_1, c_2, \ldots, c_k)) = 0.$$

Proposition 2.3. [5] *The k-th degree characteristic polynomial $x_{Q_k}(\lambda)$ of the generalized k-Fibonacci matrix $Q_k(c_1, c_2, \ldots, c_k)$ in (6) is given by*

$$(8) \qquad x_{Q_k}(\lambda) = \lambda^k - \sum_{i=1}^{k} c_i \lambda^{k-i}.$$

The set of all eigenvalues of A is denoted by $\sigma(A)$ and called the *spectrum* of A; the nonnegative real number $\rho(A) = max\{|\lambda| : \ \lambda \in \sigma(A)\}$ is called *spectral radius* of A. Recall that a matrix A with nonnegative entries is said to be *primitive*, if A is irreducible and has only one eigenvalue of maximum modulus, [4, Definition 8.5.0].

Here, q_{ij} denotes the ij-th entry of $Q_k(c_1, c_2, \ldots, c_k)$, for $1 \le i, j \le k$, and $\rho(Q_k) \equiv \rho(Q_k(c_1, c_2, \ldots, c_k))$ the spectral radius of $Q_k(c_1, c_2, \ldots, c_k)$; since q_{ij} are nonnegative real numbers according to [4, Theorem 8.1.22] we can write the following bounds

$$\ell_1 \equiv \min_{1\le j\le k} \sum_{i=1}^{k} q_{ij} \le \rho(Q_k) \le \max_{1\le j\le k} \sum_{i=1}^{k} q_{ij} \equiv w_1,$$

$$\ell_2 \equiv \min_{1\le i\le k} \sum_{j=1}^{k} q_{ij} \le \rho(Q_k) \le \max_{1\le i\le k} \sum_{j=1}^{k} q_{ij} \equiv w_2,$$

where

$$\ell_1 \equiv \min_{1\le j\le k} \sum_{i=1}^{k} q_{ij} = \min\{\min_{1\le j\le k-1}\{1+c_j\},\ c_k\},$$

$$w_1 \equiv \max_{1\le j\le k} \sum_{i=1}^{k} q_{ij} = \max\{\max_{1\le j\le k-1}\{1+c_j\},\ c_k\},$$

$$\ell_2 \equiv \min_{1\le i\le k} \sum_{j=1}^{k} q_{ij} = \min\{1,\ \sum_{j=1}^{k} c_j\},$$

$$w_2 \equiv \max_{1\le i\le k} \sum_{j=1}^{k} q_{ij} = \max\{1,\ \sum_{j=1}^{k} c_j\}.$$

Since ℓ_1, ℓ_2, w_1, w_2 are positive real numbers the above relations yield

$$\max\{\ell_1, \ell_2\} \le \rho(Q_k) \le \min\{w_1, w_2\}. \tag{9}$$

Since (8) leads to

$$det(Q_k(c_1, c_2, \ldots, c_k)) = (-1)^k(-c_k) = (-1)^{k+1}c_k, \tag{10}$$

and $c_1, c_2, \ldots, c_k$ are nonnegative real numbers with $c_1 \neq 0$, from (10) it is obvious that $Q_k(c_1, c_2, \ldots, c_k)$ is a nonsingular matrix if and only if $c_k \neq 0$, and then all the eigenvalues of $Q_k(c_1, c_2, \ldots, c_k)$ are nonzero. In the following proposition, the characteristic polynomial of the generalized k,m-Fibonacci matrix $R_{k,m}(c_1, c_2, \ldots, c_k)$ is formulated.

Proposition 2.4. *The $(k+m)$-th degree characteristic polynomial $x_{R_{k,m}}(\lambda)$ of the generalized k,m-Fibonacci matrix $R_{k,m}(c_1, c_2, \ldots, c_k)$ in (7) is given by*

$$x_{R_{k,m}}(\lambda) = \lambda^{k+m} - \sum_{i=1}^{k} c_i \lambda^{k-i}. \tag{11}$$

The characteristic polynomial in (11) for $c_1 = c_2 = \ldots = c_k = 1$ has been investigated in [1, Proposition 5]; Proposition 2.4 can been proved using the same statements as in [1, Proposition 5].

Since (11) leads to

$$det\,(R_{k,m}(c_1, c_2, \ldots, c_k)) = (-1)^{k+m}(-c_k) = (-1)^{k+m+1}c_k, \tag{12}$$

we conclude that $R_{k,m}(c_1, c_2, \ldots, c_k)$ is a nonsingular matrix if and only if $c_k \neq 0$. In this case, all the eigenvalues of $R_{k,m}(c_1, c_2, \ldots, c_k)$ are nonzero.

Let r_{ij} denote the ij-th entry of $R_{k,m}(c_1, \ldots, c_k)$, for $1 \leq i, j \leq k+m$, and $\rho(R_{k,m}) \equiv \rho(R_{k,m}(c_1, c_2, \ldots, c_k))$ the spectral radius of $R_{k,m}(c_1, c_2, \ldots, c_k)$; since r_{ij} are nonnegative real numbers and

$$\begin{aligned}
&\min_{1\leq j\leq k+m} \sum_{i=1}^{k+m} r_{ij} = \min\{1, \min_{1\leq j\leq k-1}\{1+c_j\}, c_k\} = \min\{1,\, c_k\} \equiv \tilde{\ell}_1, \\
&\max_{1\leq j\leq k+m} \sum_{i=1}^{k+m} r_{ij} = \max\{1, \max_{1\leq j\leq k-1}\{1+c_j\}, c_k\} \\
&= \max\{\max_{1\leq j\leq k-1}\{1+c_j\}, c_k\} \equiv \tilde{w}_1, \\
&\min_{1\leq i\leq k+m} \sum_{j=1}^{k+m} r_{ij} = \min\{1, \sum_{j=1}^{k} c_j\} \equiv \tilde{\ell}_2, \\
&\max_{1\leq i\leq k+m} \sum_{j=1}^{k+m} r_{ij} = \max\{1, \sum_{j=1}^{k} c_j\} \equiv \tilde{w}_2,
\end{aligned}$$

according to [4, Theorem 8.1.22], we derive

$$\tilde{\ell}_1 \leq \rho(R_{k,m}) \leq \tilde{w}_1, \tag{13}$$

$$\tilde{\ell}_2 \leq \rho(R_{k,m}) \leq \tilde{w}_2. \tag{14}$$

Since $\tilde{\ell}_1, \tilde{\ell}_2, \tilde{w}_1, \tilde{w}_2 > 0$, combining (13) and (14) we can write

$$\max\{\tilde{\ell}_1, \tilde{\ell}_2\} \leq \rho(R_{k,m}) \leq \min\{\tilde{w}_1, \tilde{w}_2\}. \tag{15}$$

For the integers $k \geq 2$ and $m > 0$, since the entries of the matrix

$$(I_{k+m} + R_{k,m}(c_1, c_2, \ldots, c_k))^{k+m-1}$$

are positive real numbers, the generalized k, m-Fibonacci matrix $R_{k,m}(c_1, c_2, \ldots, c_k)$ is an irreducible matrix, [4, Lemma 8.4.1]; it follows that the spectral radius $\rho(R_{k,m})$ is a positive, simple (without multiplicity) eigenvalue of $R_{k,m}(c_1, c_2, \ldots, c_k)$, [4, Theorem 8.4.4].

Additionally, the entries of $R_{k,m}(c_1, c_2, \ldots, c_k)^{(k+m)^2-2(k+m)+2}$ are positive real numbers, thus $R_{k,m}(c_1, c_2, \ldots, c_k)$ is a primitive matrix [4, Corollary 8.5.9], i.e., $\rho(R_{k,m})$ is the unique eigenvalue with maximum modulus [4, Definition 8.5.0]. Hence, in the following, we denote $\lambda_1, \lambda_2, \ldots, \lambda_{k+m-1}, \rho(R_{k,m})$ all the distinct eigenvalues of $R_{k,m}(c_1, c_2, \ldots, c_k)$, for which the following inequality holds

$$0 < \ |\lambda_j| < \rho(R_{k,m})\,; \ \ j = 1, 2, \ldots, k+m-1. \tag{16}$$

In the following, we may rewrite the terms of the Fibonacci sequence

$$\left(f_n^{\{k,m\}}(c_1,c_2,\dots,c_k)\right)_{n=1,2,\dots}$$

such that some initial terms can be defined by negative indexed. To this end, we use the *Dirac delta function* (or δ function), which is denoted by $\delta_{n-j}=\begin{cases}0, & n\neq j\\ 1, & n=j\end{cases}$, and the *Heaviside step function* $u_{n-j}=\begin{cases}0, & n<j\\ 1, & n\geq j\end{cases}$.

Consider that, the first $k+m$ negative indexed terms are equal to zero

$$f_{-(k+m-1)}=\dots=f_{-1}=f_0=0, \tag{17}$$

then the n-th term f_n of the generalized k,m-step Fibonacci sequence $\left(f_n^{\{k,m\}}(c_1,c_2,\dots,c_k)\right)_{n=1,2,\dots}$, which is formulated in the following proposition, follows immediately from (2) and (3).

Proposition 2.5. *Let $c_1,c_2,\dots,c_k$ be the given nonnegative real numbers with $c_1\neq 0$, and the given integers $k\geq 2$, $m\geq 0$. For all $n=1,2,\dots$, the n-th term f_n of the generalized k,m-step Fibonacci sequence is given by the following recurrence relation*

$$f_n=\sum_{i=1}^{k}c_if_{n-m-i}+\sum_{i=1}^{k+m}\delta_{n-i}-\sum_{j=1}^{k-1}c_j\sum_{i=1}^{k-j}\delta_{n-m-j-i}, \tag{18}$$

with initial values in (17).

By the recurrence relation in (18) and the initial values in (17) for $m=0$, the n-th term f_n of the k-step Fibonacci sequence $\left(f_n^{\{k,0\}}(c_1,c_2,\dots,c_k)\right)_{n=1,2,\dots}$ is given by

$$f_n=\sum_{i=1}^{k}c_if_{n-i}+\sum_{i=1}^{k}\delta_{n-i}-\sum_{j=1}^{k-1}c_j\sum_{i=1}^{k-j}\delta_{n-j-i}, \tag{19}$$

with initial values

$$f_{-(k-1)}=\dots=f_{-1}=f_0=0. \tag{20}$$

Furthermore, the z-transform on both sides of (18) yields

$$F(z)=\frac{1}{z^{-(k+m)}x_{R_{k,m}}(z)}\left(z^{-1}+\dots+z^{-(k+m)}-c_1\sum_{i=1}^{k-1}z^{-(m+1+i)}\right.$$
$$\left.-c_2\sum_{i=1}^{k-2}z^{-(m+2+i)}-\dots-c_{k-2}\sum_{i=1}^{2}z^{-(k+m+i-2)}-c_{k-1}z^{-(k+m)}\right). \tag{21}$$

From (21) it is worth noting that the poles of $F(z)$ are the eigenvalues of $R_{k,m}(c_1,c_2,\dots,c_k)$, which are simple (distinct) and the complex eigenvalues are conjugate; furthermore, the degree of the polynomial of denominator of $F(z)$ is greater or equal than the degree of the polynomial of the numerator, thus, the partial-fraction decomposition of (21) is given by

$$F(z) = 1 - \sum_{i=1}^{k-1} c_i + \frac{a}{1-\rho(R_{k,m})z^{-1}} + \sum_{j=1}^{k+m-1} \frac{a_j}{1-\lambda_j z^{-1}}, \tag{22}$$

where $a, \rho(R_{k,m}) \in \mathbb{R}$, and the others coefficients a_j are complex or real numbers.

In the following proposition, we are able to present the closed formula of the terms of the sequence $\left(f_n^{\{k,m\}}(c_1, c_2, \ldots, c_k)\right)_{n=1,2,\ldots}$, which depends on the spectral radius $\rho(R_{k,m})$ of

$$R_{k,m}(c_1, c_2, \ldots, c_k)$$

and the others eigenvalues λ_j of $R_{k,m}(c_1, c_2, \ldots, c_k)$.

Proposition 2.6. *Let $\lambda_1, \lambda_2, \ldots, \lambda_{k+m-1}, \rho(R_{k,m})$ be the eigenvalues of the generalized k,m-Fibonacci matrix $R_{k,m}(c_1, c_2, \ldots, c_k)$ and the fixed integers k,m, with $k \geq 2$, $m > 0$. The n-th term f_n of the generalized k,m-step Fibonacci sequence $\left(f_n^{\{k,m\}}(c_1, c_2, \ldots, c_k)\right)_{n=1,2,\ldots}$ is given by*

$$f_n = a\,(\rho(R_{k,m}))^n + \sum_{j=1}^{k+m-1} a_j (\lambda_j)^n, \tag{23}$$

where a, a_j, for all $j = 1, 2, \ldots, k+m-1$, are the determined coefficients of the partial-fraction decomposition in (22).

Proof. For all $n = 1, 2, \ldots$ the inverse z-transformation on both sides of (22) yields

$$f_n = \left(1 - \sum_{i=1}^{k-1} c_i\right)\delta_n + a\,(\rho(R_{k,m}))^n u_n + \sum_{j=1}^{k+m-1} a_j (\lambda_j)^n u_n.$$

The closed formula of f_n in (23) follows from the above equation and the definitions of δ and Heaviside step functions. □

The spectral radius $\rho(R_{k,m})$ of $R_{k,m}(c_1, c_2, \ldots, c_k)$ in (7) is a characteristic quantity, which appears in (23) and its significance is presented in the following theorem.

Theorem 2.7. *For the fixed integers k,m, with $k \geq 2$, $m > 0$, the positive numbers f_n in (23) satisfy the limit properties*

$$\lim_{n\to\infty} \frac{f_{n+1}}{f_n} = \rho(R_{k,m}), \tag{24}$$

$$\lim_{n\to\infty} \sqrt[n]{f_n} = \rho(R_{k,m}), \tag{25}$$

where $\rho(R_{k,m})$ is the spectral radius of $R_{k,m}(c_1, c_2, \ldots, c_k)$ in (7).

Proof. Consider that in (22) the polar form of the determined coefficients a_j is denoted by $a_j = |a_j|e^{i\theta_j}$, and the polar form of the eigenvalues (except the spectral radius) is given by $\lambda_j = |\lambda_j| e^{i\omega_j}$, for all $j = 1, 2, \ldots, k+m-1$. The substitution of a_j, λ_j from their polar forms in (23) yields

$$\begin{aligned} f_n &= a\,(\rho(R_{k,m}))^n + \sum_{j=1}^{k+m-1} |a_j| e^{i\theta_j} (|\lambda_j| e^{i\omega_j})^n \\ &= a\,(\rho(R_{k,m}))^n + \sum_{j=1}^{k+m-1} |a_j|\, |\lambda_j|^n e^{i(\theta_j + n\omega_j)}. \end{aligned} \tag{26}$$

Using (26) and the property of the spectral radius $\rho(R_{k,m}) > 0$ from (16), we can write

$$\lim_{n\to\infty} \frac{f_{n+1}}{f_n} = \lim_{n\to\infty} \frac{(\rho(R_{k,m}))^{n+1} \,(a + \sum_{j=1}^{k+m-1} |a_j| \left|\frac{\lambda_j}{\rho(R_{k,m})}\right|^{n+1} e^{i(\theta_j + (n+1)\omega_j)})}{(\rho(R_{k,m}))^{n} \,(a + \sum_{j=1}^{k+m-1} |a_j| \left|\frac{\lambda_j}{\rho(R_{k,m})}\right|^{n} e^{i(\theta_j + n\omega_j)})}. \tag{27}$$

Since the inequality in (16) implies $\left|\frac{\lambda_j}{\rho(R_{k,m})}\right| < 1$, for $j = 1, 2, \ldots, k+m-1$, and the sequence $\left(e^{i(\theta_j + n\omega_j)}\right)_{n=1,2,\ldots}$ is bounded as the sum of the bounded sequences $(\cos(\theta_j + n\omega_j))_{n=1,2,\ldots}$ as well as $(\sin(\theta_j + n\omega_j))_{n=1,2,\ldots}$, it is obvious that

$$\lim_{n\to\infty} \left|\frac{\lambda_j}{\rho(R_{k,m})}\right|^{n+1} e^{i(\theta_j + (n+1)\omega_j)} = \lim_{n\to\infty} \left|\frac{\lambda_j}{\rho(R_{k,m})}\right|^{n} e^{i(\theta_j + n\omega_j)} = 0. \tag{28}$$

Thus, the validity of (24) follows from (27) and (28).

Furthermore, it is well-known that for a sequence $(\alpha_n)_{n=1,2,\ldots}$ of nonzero complex numbers, if $\lim\limits_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} = \alpha$, then $\lim\limits_{n\to\infty} \sqrt[n]{|\alpha_n|} = \alpha$, [9, Chapter 1], whereby it is evident that for the generalized k,m-step Fibonacci sequence $\left(f_n^{\{k,m\}}(c_1, c_2, \ldots, c_k)\right)_{n=1,2,\ldots}$ of the positive real numbers ($f_n > 0$), the equality (25) follows immediately from (24). □

Notice that, for $m = 0$, the characteristic polynomials $x_{R_{k,m}}(\lambda)$ in (11) and $x_{Q_k}(\lambda)$ in (8) coincide; hence the z-transform on both sides of (19) yields

$$F(z) = 1 - \sum_{i=1}^{k-1} c_i + \frac{a}{1 - \rho(Q_k) z^{-1}} + \sum_{j=1}^{k-1} \frac{a_j}{1 - \lambda_j z^{-1}}, \tag{29}$$

where λ_j denotes all the others eigenvalues of the generalized k-Fibonacci matrix $Q_k(c_1, c_2, \ldots, c_k)$ except its spectral radius $\rho(Q_k)$, and $a, \rho(Q_k) \in \mathbb{R}$, the coefficients a_j are complex or real numbers.

The inverse z-transform on both sides of (29) for all $n = 1, 2, \ldots$ and the definitions of δ and Heaviside step functions follow the closed formula of the n-th term f_n of the generalized k-step Fibonacci sequence, which is formulated as

$$f_n = a\,(\rho(Q_k))^n + \sum_{j=1}^{k+m-1} a_j(\lambda_j)^n. \tag{30}$$

In the following, the limiting properties of the generalized k-step Fibonacci sequence $\left(f_n^{\{k,0\}}(c_1, c_2, \ldots, c_k)\right)_{n=1,2,\ldots}$ are presented.

Theorem 2.8. *Let $\rho(Q_k)$ be the spectral radius of the generalized k-Fibonacci matrix $Q_k(c_1, c_2, \ldots, c_k)$ in (6) with $k \geq 2$. The positive numbers f_n in (30) satisfy the limit properties*

$$\lim_{n\to\infty} \frac{f_{n+1}}{f_n} = \rho(Q_k), \qquad \lim_{n\to\infty} \sqrt[n]{f_n} = \rho(Q_k). \tag{31}$$

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