HOMOGENEOUS DIRECT PRODUCT OF GENERALIZED MULTIAUTOMATA

P. RAČKOVÁ

Department of Mathematics and Physics, University of Defence, Kounicova 65, 662 10 Brno, Czech Republic
pavlina.rackova@unob.cz

Abstract. In several papers direct products of multiautomata were studied. While there are no problems with the heterogeneous product some kind of a generalized multiautomaton had to be introduced to overcome problems with the homogeneous product. In the article this concept is extended to make the situation clearer. It will be shown that the homogeneous product of such generalized multiautomata each time exists. As any multiautomaton naturally induces a generalized multiautomaton this fact enables us to define likewise the direct homogeneous product of multiautomata with the result being the generalized multiautomaton.

1. Introduction

The concept of direct product plays an important role in many parts of algebra. For example, it is easy to define both homogeneous and heterogeneous product for automata. Let us shortly recall the corresponding definitions and results.

Definition 1. Let S (*set of states*) and A (*set of input symbols*) be nonempty sets and A^* be a free monoid of words of the alphabet A with the operation of concatenation and an empty word e as the neutral element.
An *automaton without output* is a triple $\mathbb{A} = (S, A, \delta)$, where a mapping $\delta\colon S \times A^* \to S$ satisfies the following conditions:

1) $\delta(s, e) = s$ for any state $s \in S$,
2) $\delta(s, ab) = \delta\big(\delta(s, a), b\big)$ for any state $s \in S$ and any pair of words $a, b \in A^*$.

Mapping δ is called a *transition function*.

Two kinds of direct products are defined in automata theory—homogeneous and heterogeneous product. In case of the homogeneous product the input alphabet is the same for all automata and their product. On the other hand, in case of

Received: 6 September 2015; Accepted: 1 December 2015
Key words and phrases: Hypergroupoid, hypergroup, generalized multiautomaton, direct homogeneous and heterogeneous products.
2010 AMS Mathematics Subject Classification: 20N20, 20M35, 20K25.
This paper has been supported by ZRO K215 Project.

the heterogeneous product input alphabets of individual automata are different in general and the input alphabet of their product is equal to the Cartesian product of the input alphabets.

Definition 2. Let $\mathbb{A}_i = (S_i, A, \delta_i)$, where $i = 1, \dots, k$, be automata with the same input alphabet A. The *homogeneous direct product of the automata* $\mathbb{A}_1, \mathbb{A}_2, \dots, \mathbb{A}_k$ is an automaton $\mathbb{A} = \mathbb{A}_1 \times \mathbb{A}_2 \times \dots \times \mathbb{A}_k = (S, A, \delta)$, where $S = S_1 \times \dots \times S_k$ and the function $\delta\colon S \times A^* \to S$ is defined by the formula

$$\delta\big((s_1, \dots, s_k), a\big) = \big(\delta_1(s_1, a), \dots, \delta_k(s_k, a)\big) \quad \text{for any } s_i \in S_i, a \in A^*.$$

Definition 3. Let $\mathbb{A}_i = (S_i, A_i, \delta_i)$, where $i = 1, \dots, k$, be automata with possibly different input alphabets A_i. The *heterogeneous direct product of the automata* $\mathbb{A}_1, \mathbb{A}_2, \dots, \mathbb{A}_k$ is an automaton $\mathbb{A} = \mathbb{A}_1 \otimes \mathbb{A}_2 \otimes \dots \otimes \mathbb{A}_k = (S, A, \delta)$, where $S = S_1 \times \dots \times S_k$, $A = A_1 \times \dots \times A_k$ and the function $\delta\colon S \times A^* \to S$ is defined as follows: If

$$s = (s_1, \dots, s_k) \in S = S_1 \times \dots \times S_k,$$
$$a = (a_1^{(1)}, \dots, a_k^{(1)}) \dots (a_1^{(m)}, \dots, a_k^{(m)}) \in A^* = (A_1 \times \dots \times A_k)^*,$$

then

$$\delta(s, a) = \big(\delta_1(s_1, a_1), \dots, \delta_k(s_k, a_k)\big), \quad \text{where}$$
$$a_1 = a_1^{(1)} a_1^{(2)} \dots a_1^{(m)} \in A_1^*, \dots,\ a_k = a_k^{(1)} a_k^{(2)} \dots a_k^{(m)} \in A_k^*.$$

It is easy to see that an automaton is really obtained by the constructions described in Definition 2 and 3.

2. Multiautomata

In connection with the study of hyperstructures the concept of multiautomaton, where the input alphabet is enriched with a hyperoperation and a so called generalized mixed associativity condition holds, was introduced, see, e.g., [5, 6]. Prior to the definition of multiautomaton we shortly sum up necessary concepts from hyperstructure theory needed in the following exposition.

Let H be a nonempty set and $\mathcal{P}^*(H)$ be a set of all nonempty subsets of H (potential set of H). A mapping $*\colon H \times H \to \mathcal{P}^*(H)$ is called *binary hyperoperation.*

The set H with a binary hyperoperation $*$, i.e. the pair $(H, *)$, is called *hypergroupoid* and the set $x * y$ is the *hyperproduct* of x and y. A hypergroupoid $(H, *)$ is called *commutative* if $a * b = b * a$ for any $a, b \in H$.

For any $X, Y \in H^*$ and $y \in H$ we set

$$X * Y = \bigcup_{(x,y) \in X \times Y} x * y \quad \text{and} \quad y * X = \{y\} * X.$$

A hypergroupoid $(H, *)$ is called *semihypergroup* if it is associative, i.e. $(x*y)*z = x * (y * z)$ for any $x, y, z \in H$.

A hypergroupoid $(H, *)$ is called *quasihypergroup* if $a * H = H = H * a$ for each $a \in H$ (reproduction axiom).

A semihypergroup $(H, *)$ is called *hypergroup* if the reproduction axiom is satisfied.

Definition 4. Let S be a nonempty set, $\mathbb{H} = (H, *)$ be a hypergroupoid and $\delta\colon S \times H \to S$ be a mapping satisfying the condition (so called Generalized Mixed Associativity Condition (GMAC))

$$\delta\big(\delta(s, a), b\big) \in \delta(s, a * b) \tag{1}$$

for every triple $(s, a, b) \in S \times H \times H$, where $\delta(s, a * b) = \{\delta(s, x), x \in a * b\}$.

The triple $\mathbb{M} = (S, \mathbb{H}, \delta)$ is called *multiautomaton* with the state set S and the input hypergroupoid $\mathbb{H}$. The mapping $\delta\colon S \times H \to S$ is called *the transition function of multiautomaton* $\mathbb{M}$.

Since any groupoid is also a hypergroupoid, the concept of multiautomaton can be viewed as the generalization of automaton.

It is natural to make an attempt to carry over direct products to multiautomata. No problem occurs as to the heterogeneous product.

Definition 5. Let $\mathbb{M}_i = (S_i, \mathbb{H}_i, \delta_i)$, $\mathbb{H}_i = (H_i, *_i)$, $i = 1, 2, \ldots, k$, be multiautomata. We denote $S = \prod\limits_{i=1}^{k} S_i$, $H = \prod\limits_{i=1}^{k} H_i$ and define the hyperoperation $*$ on H by the formula

$$(a_1, \ldots, a_k) * (b_1, \ldots, b_k) = \prod_{i=1}^{k} a_i *_i b_i.$$

For $s = (s_1, \ldots, s_k) \in S$ and $a = (a_1, \ldots, a_k) \in H$ we set

$$\delta(s, a) = \big(\delta_1(s_1, a_1), \ldots, \delta_k(s_k, a_k)\big).$$

The triple $\mathbb{M} = (S, \mathbb{H}, \delta)$, where $\mathbb{H} = (H, *)$, is called *the direct heterogeneous product of multiautomata* $\mathbb{M}_1, \ldots, \mathbb{M}_k$.

For the proof that δ satisfies the Generalized Mixed Associativity Condition see [5, 6].

The straightforward way fails in the case of the homogeneous product. If we assume that all multiautomata $\mathbb{M}_i = (S_i, \mathbb{H}, \delta_i)$, $i = 1, 2, \ldots, k$, have the same input hypergroup $\mathbb{H} = (H, *)$ and analogously to Definition 2 we set $S = \prod\limits_{i=1}^{k} S_i$ and $\delta\big((s_1, \ldots, s_k), a\big) = \big(\delta_1(s_1, a), \ldots, \delta_k(s_k, a)\big)$, the triple $\mathbb{M} = (S, H, \delta)$ need not be a multiautomaton as the mapping $\delta\colon S \times H \to S$ need not satisfy GMAC. See, e.g., [5, 6], where a detailed analysis is given and a counterexample is presented.

To overcome this restriction, generalized multiautomata were introduced in [5] and their direct homogeneous product was defined. In the following section this concept will be extended to enable a better analysis of the mentioned problem with the homogeneous product.

3. **Generalized Multiautomata and their Direct Homogeneous Product**

Unlike the multiautomaton where a new state is assigned to each state and each input, in the case of the generalized multiautomaton, a set of states will be assigned to each set of states and each set of inputs.

Definition 6. Let S be a nonempty set, $\mathbb{H} = (H, *)$ be a hypergroupoid and $\delta\colon \mathcal{P}^*(S) \times \mathcal{P}^*(H) \to \mathcal{P}^*(S)$ be a mapping satisfying the condition (so called Generalized Mixed Associativity Condition (GMAC))

$$\delta\big(\delta(X, A), B\big) \subset \delta(X, A * B) \tag{2}$$

for every triple $(X, A, B) \in \mathcal{P}^*(S) \times \mathcal{P}^*(H) \times \mathcal{P}^*(H)$.

The triple $\mathbb{M} = (S, \mathbb{H}, \delta)$ is called *generalized multiautomaton* with the state set S and the input hypergroupoid $\mathbb{H}$. The mapping $\delta\colon \mathcal{P}^*(S) \times \mathcal{P}^*(H) \to \mathcal{P}^*(S)$ is called *the transition function of generalized multiautomaton* $\mathbb{M}$.

Comment 1. Definition 6 is the generalization of Definition III.15 from [5] where the transition function δ was the mapping of $S \times \mathcal{P}^*(H)$ to $\mathcal{P}^*(S)$ and two additional conditions had to be satisfied: δ is monotonic with respect to the second argument and $\delta(s, \{a\})$ contains only one element for any $s \in S$ and $a \in H$.

Example 1. Assume that $\mathbb{M} = (S, \mathbb{H}, \delta)$, where $\mathbb{H} = (H, *)$, is a multiautomaton. Then $\mathbb{M}$ determines in a natural way a generalized multiautomaton $\overline{\mathbb{M}}$ with the same state set S and the input hypergroupoid $\mathbb{H}$ and the transition function $\bar{\delta}$ defined in the following way:

$$\bar{\delta}(X, A) = \bigcup_{\substack{s \in X \\ a \in A}} \{\delta(s, a)\}$$

for $X \in \mathcal{P}^*(S)$ and $A \in \mathcal{P}^*(H)$.

Indeed, for any $X \in \mathcal{P}^*(S)$ and $A, B \in \mathcal{P}^*(H)$ we get

$$\begin{aligned}\bar{\delta}\big(\bar{\delta}(X, A), B\big) = \bar{\delta}\Big(\bigcup_{\substack{s \in X \\ a \in A}} \{\delta(s, a)\}, B\Big) = \bigcup_{\substack{s \in X \\ a \in A \\ b \in B}} \big\{\delta\big(\delta(s, a), b\big)\big\} \subset \\ \subset \bigcup_{\substack{s \in X \\ a \in A \\ b \in B}} \bar{\delta}(\{s\}, a * b) = \bigcup_{\substack{s \in X \\ c \in A * B}} \{\delta(s, c)\} = \bar{\delta}(X, A * B),\end{aligned}$$

hence GMAC (2) is satisfied and $\overline{\mathbb{M}} = (S, \mathbb{H}, \bar{\delta})$ is the generalized multiautomaton.

For arbitrary sets S_i, $i = 1, \dots, k$, and $X \subset S = S_1 \times \dots \times S_k$ we denote $\mathrm{Pr}_i(X)$ the projection of X on S_i, i.e.

$$\begin{aligned}\mathrm{Pr}_i(X) = \{x \in S_i \mid\ & (a_1, \dots, a_{i-1}, x, a_{i+1}, \dots, a_k) \in X \text{ for some} \\ & a_1 \in S_1, \dots, a_{i-1} \in S_{i-1}, a_{i+1} \in S_{i+1}, \dots, a_k \in S_k\}.\end{aligned}$$

Evidently, $X \subset \mathrm{Pr}_1(X) \times \dots \times \mathrm{Pr}_k(X)$. Furthermore, if $X_i \subset S_i$, $i = 1, \dots, k$, and $X = X_1 \times \dots \times X_k$, then $\mathrm{Pr}_i(X) = X_i$, thus $X = \mathrm{Pr}_1(X) \times \dots \times \mathrm{Pr}_k(X)$.

Theorem 1. *Let* $\mathbb{M}_i = (S_i, \mathbb{H}, \delta_i)$, $i = 1, \dots, k$, *be generalized multiautomata with the same input hypergroupoid* $\mathbb{H} = (H, *)$. *Let us denote* $S = \prod_{i=1}^{k} S_i$ *and define a mapping* $\delta \colon \mathcal{P}^*(S) \times \mathcal{P}^*(H) \to \mathcal{P}^*(S)$ *as follows:*

$$\delta(X, A) = \delta_1(\mathrm{Pr}_1(X), A) \times \cdots \times \delta_k(\mathrm{Pr}_k(X), A)$$

for $X \in \mathcal{P}^*(S)$ *and* $A \in \mathcal{P}^*(H)$.

Then the triple $\mathbb{M} = (S, \mathbb{H}, \delta)$ *is the generalized multiautomaton.*

Proof. We have to verify that GMAC (2) is satisfied. For any $X \in \mathcal{P}^*(S)$ and $A, B \in \mathcal{P}^*(H)$ we have

$$\begin{aligned}
\delta\big(\delta(X, A), B\big) &= \delta\Big(\prod_{i=1}^{k} \delta_i\big(\mathrm{Pr}_i(X), A\big), B\Big) = \\
&= \prod_{j=1}^{k} \delta_j\Big(\mathrm{Pr}_j\Big(\prod_{i=1}^{k} \delta_i\big(\mathrm{Pr}_i(X), A\big)\Big), B\Big) = \\
&= \prod_{j=1}^{k} \delta_j\Big(\delta_j\big(\mathrm{Pr}_j(X), A\big), B\Big) \subset \\
&\subset \prod_{j=1}^{k} \delta_j\big(\mathrm{Pr}_j(X), A * B\big) = \\
&= \delta(X, A * B).
\end{aligned}$$

Therefore, $\mathbb{M} = (S, \mathbb{H}, \delta)$ is the generalized multiautomaton. □

Definition 7. The multiautomaton $\mathbb{M} = (S, \mathbb{H}, \delta)$ from Theorem 1 is called *the direct homogeneous product* of generalized multiautomata $\mathbb{M}_i = (S_i, \mathbb{H}, \delta_i)$, $i = 1, \dots, k$.

Assume that $\mathbb{M}_i = (S_i, \mathbb{H}, \delta_i)$, $i = 1, \dots, k$, are multiautomata. According to Example 1 they determine generalized multiautomata $\overline{\mathbb{M}}_i = (S_i, \mathbb{H}, \bar{\delta}_i)$. The direct homogeneous product of these can be viewed as the direct homogeneous product of multiautomata $\mathbb{M}_i$.

Therefore, the direct homogeneous product of multiautomata is in general the generalized multiautomaton.

References

[1] Z. Bavel, *The source as a tool in automata.* Inform. Control. **18** (1971), 140–155.

[2] P. Corsini, V. Leoreanu, *Applications of Hyperstructure Theory.* Kluwer AP, Dordrecht, Boston, London 2003, 322 pp.

[3] J. Chvalina, L. Chvalinová, *State hypergroups of automata.* Acta Math. et Inf. Univ. Ostraviensis **4** (1996), 105–120.

[4] J. Chvalina, J. Moučka, P. Račková, *Direct product of multiautomata as a tool for modelling of parallel processes.* XXIII$^{\mathrm{d}}$ International Colloquium on the Acquisition Process Management, Brno, 2005, CD-ROM, 7 p.

[5] P. Račková, *Actions of semihypergroups and modelling of hypergroups by integral operators.* PhD. Thesis, Faculty of Science, Palacký University, Olomouc 2008, Czech Rep., 73 pp. (in Czech)

[6] J. Chvalina, P. Račková, *Modified homogeneous product of multiautomata.* In 6^{th} International Conference Aplimat, Bratislava, 2007, Part III, 83–88.

[7] P. Račková, *Direct products of multiautomata.* Proceedings of International Conference Presentation of Mathematics '05, Liberec, 2005, 4 p. (in Czech)

[8] P. Račková, Š. Hošková, *Heterogeneous direct product of multiautomata.* 4^{th} Conference on Mathematics and Physics at Technical Universities, Brno 2005, CD ROM, 61–65.