SEVERAL ASPECTS OF GENERALIZING ONE CONSTRUCTION OF HYPERSTRUCTURES FROM QUASI-ORDERED SEMIGROUPS

M. NOVÁK AND Š. KŘEHLÍK

Faculty of Electrical Engineering and Communication, Brno University of Technology
Technická 10, 616 00 Brno, Czech Republic
novakm@feec.vutbr.cz, xkrehl02@stud.feec.vutbr.cz

Abstract. EL–hyperstructures are hyperstructures constructed from single-valued quasi-ordered semigroups. For some kinds of sets it is difficult to find a meaningful single-valued associative operation which could be used as a basis for constructing the EL–hyperstructure. In this paper we use the systematic approach to define it. We focus on multicomponent sets and briefly mention the n–ary context of the construction.

1. Introduction

The study of links between hyperstructures and orderings is one of the classical topics of the hyperstructure theory connected with names such as Nieminen, Corsini, Rosenberg, Krasner, Mittas, Davvaz, Leoreanu or Chvalina and their works published in 1960s to 1990s. Chvalina [6] studied various aspects of this link. One of these is the relation between quasi-ordered semigroups and hyperstructures. Theoretical aspects of one particular construction in this area, resulting in what is known as *EL–hyperstructures*, have been studied by Novák in [19, 21], extended to the n–ary context in [11] and studied in detail in [20]. However, the range of possible applications of the construction is such that a systematic study of its limitations is needed (for a variety of uses see Examples 3.6 – 3.9 in this paper).

In fact the main motivation for this paper lies in contexts such as the ones described in Section 3, Example 3.8 or Example 3.9. In the particular context of these examples it is various aspects of electrical engineering (such as measurement of broadband structures in time domain, various high frequency and microwave techniques, nontraditional measurement of microwave structures in time domain, etc.) or study of functions modelling growth or increase of certain ecosystems that can be described by the results included in this paper.

When defining basic concepts of the hyperstructure theory we use definitions included in [2, 3]. In the n–ary hyperstructure context we follow the approach

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of [17]. For a deeper insight in the issue of hyperstructures and their connection to n–ary relations cf. e.g. [4, 5]; for ideas concerning the problems connected to n–ary groups cf. e.g. [12, 13].

2. **Preliminaries**

In this paper we use standard notions and concepts of hyperstructure theory such as *hypergroupoid*, *(semi)hypergroup*, *transposition hypergroup* or *join space.*

We link concepts of the hyperstructure theory to concepts and notions of the theory of ordered structures such as *quasi-ordering*, i.e. a reflexive and transitive relation, or *partial ordering*, i.e. a reflexive, antisymmetric and transitive relation. We discuss *quasi*– or *partially ordered (semi)groups*, i.e. (semi)groups with the property $a \leq b$ implies $a \cdot c \leq b \cdot c$ and $c \cdot a \leq c \cdot b$, where $(a, b, c) \in S^3$.

In the last section of the paper we also work with the generalization of some basic concepts of the hyperstructure theory. Theorems 5.1 and 5.2 make use of the following three definitions included in [10].

Definition 2.1. [10] Let H be a non-empty set and f be a mapping $f : H \times H \to P^*(H)$, where $P^*(H)$ is the set of all non-empty subsets of H. Then f is called a binary hyperoperation of H. We denote by H^n the cartesian product $H \times \ldots \times H$, where H appears n times. An element of H^n will be denoted by $(x_1, \ldots, x_n)$, where $x_i \in H$ for any i with $1 \leq i \leq n$. In general, a mapping $f : H^n \to P^*(H)$ is called an n–ary hyperoperation and n is called the arity of hyperoperation. Let f be an n–ary hyperoperation on H and $A_1, \ldots, A_n$ subsets of H. We define

$$f(A_1, \ldots, A_n) = \cup\{f(x_1, \ldots, x_n)|x_i \in A_i, i = 1, \ldots, n\}.$$

We shall use the following abbreviated notation: the sequence $x_i, x_{i+1}, \ldots, x_j$ will be denoted by x_i^j. For $j < i, x_i^j$ is the empty set. In this convention

$$f(x_1, \ldots, x_i, y_{i+1}, \ldots, y_j, z_{j+1}, \ldots, z_n)$$

will be written as $f(x_1^i, y_{i+1}^j, z_{j+1}^n)$.

Definition 2.2. [10] A non-empty set H with an n–ary hyperoperation $f : H^n \to P^*(H)$ will be called an n–ary hypergroupoid and will be denoted by (H, f). An n–ary hypergroupoid (H, f) will be called an n–ary semihypergroupoid if and only if the following associative axiom holds:

$$f(x_1^{i-1}, f(x_i^{n+i-1}), x_{n+i}^{2n-1}) = f(x_1^{j-1}, f(x_j^{n+j-1}), x_{n+j}^{2n-1}) \tag{1}$$

for every $i, j \in \{1, 2, \ldots, n\}$ and $x_1, x_2, \ldots, x_{2n-1} \in H$.

Definition 2.3. [10] An n–ary semihypergroup (H, f) in which the equation

$$b \in f(a_1^{i-1}, x_i, a_{i+1}^n) \tag{2}$$

has the solution $x_i \in H$ for every $a_1, \ldots a_{i-1}, a_{i+1}, \ldots a_n, b \in H$ and $1 \leq i \leq n$, is called an n–ary hypergroup.

3. The construction

Throughout the paper we test possibilities and limitations offered by the construction discussed in this section. Notice that this construction was introduced in [6] and expanded in [21, 22]. First of all we give the construction and "set the borders" to our future considerations. Then we include some examples (both trivial and non-trivial) to demonstrate problems discussed in further sections.

Lemma 3.1. [6] *Let* $(S, \cdot, \leq)$ *be a partially ordered semigroup. Binary hyperoperation* $*: S \times S \to \mathcal{P}'(S)$ *defined by*

$$a * b = [a \cdot b)_{\leq}$$

is associative. The semihypergroup $(S, *)$ *is commutative if and only if the semigroup* $(S, \cdot)$ *is commutative.*

Lemma 3.2. [6] *The following conditions are equivalent:*

(1) *For any pair* $a, b \in S$ *there exists a pair* $c, c' \in S$ *such that* $b \cdot c \leq a$ *and* $c' \cdot b \leq a$

(2) *The associated semihypergroup* $(S, *)$ *is a hypergroup.*

Remark 3.3. With the exception of "$\Leftarrow$" of the part regarding commutativity, Lemma 3.1 is valid for quasi-ordered structures as well. Notice that condition 1 of Lemma 3.2 holds trivially in groups. The issue of semigroups not being groups and yet creating hypergroups was treated in [19].

Lemma 3.4. [22] *Let* $(H, \cdot, \leq)$ *be a quasi-ordered group and* $(H, *)$ *be the associated hypergroup. Then* $(H, *)$ *is a transposition hypergroup.*

Lemma 3.5. [21] *Let* $(H, \cdot, \leq)$ *be a non-trivial quasi-ordered group, where the relation* $\leq$ *is not the identity relation, and let* $(H, *)$ *be its associated transposition hypergroup. Then* $(H, *)$ *does not have a scalar identity.*

Thus we see that the construction, known as the *Ends lemma*, gives rise to semihypergroups, hypergroups, transposition hypergroups and join spaces yet not canonical hypergroups. Such hyperstructures have since [18] been called *EL–hyperstructures*. For further reading on this type of hyperstructures cf. e.g. [19, 20, 21].

Meaning of the above construction may be described in popular words as "everything above the product of two elements", "all descendants of two parents", "everything resulting from the fact that two elements have met", etc. It has been used (or can be used) in the study of differential / integro-differential / translation operators of various kinds [7, 8, 14], of some areas of physics or chemistry, of family relations between individuals based on tissue samples, preference relations in microeconomics [9], etc.

Example 3.6. Consider the set $\mathbb{N}$ of all natural numbers (excluding 0). Obviously $(\mathbb{N}, \cdot, \leq)$, where $\cdot$ is the usual multiplication and $\leq$ is the natural ordering of natural numbers, is a quasi-ordered semigroup. Thus if we define

$$a * b = [a \cdot b)_{\leq} = \{x \in \mathbb{N}; a \cdot b \leq x\},$$

for all $a, b \in \mathbb{N}$, then $(N, *)$ is a semihypergroup.

Example 3.7. If in the previous example we consider the set $\mathbb{R}$ of all real numbers and consider the usual addition and ordering of real numbers, then $(\mathbb{R}, +, \leq)$ is a quasi-ordered group. Thus if we define $*$ analogically as in the previous example, we get that $(\mathbb{R}, *)$ is a semihypergroup. Thanks to Lemma 3.2, 3.2 and 3.5, it is a transposition hypergroup, yet it is not a canonical hypergroup.

Example 3.8. (included in [7], used to demonstrate some results of [21]) We study the relation of hyperstructures and homogeneous second order linear differential equations

$$y'' + p(x)y' + q(x)y = 0, \tag{3}$$

such that $p \in C_+(I)$, $q \in C(I)$, where $C^k(I)$ denotes the commutative ring of all continuous real functions of one variable defined on an open interval I of reals with continuous derivatives up to order $k \geq 0$ (instead of $C^0(I)$ we write only $C(I)$), and $C_+(I)$ denotes its subsemiring of all positive continuous functions. The set of nonsingular ordinary differential equations (3) is denoted $\mathbb{A}_2$. The pair of functions p, q is denoted $[p, q]$, $D = \frac{\mathrm{d}}{\mathrm{d}x}$ and Id is the identity operator. The notation $L(p, q)$ is reserved for the differential operator $L(p, q) = D^2 + p(x)D + q(x)Id$, i.e. the notation $L(p, q)(y) = 0$ is the equation (3). The set

$$\mathbb{LA}_2(I) = \{L(p, q) : C^2(I) \to C(I); [p, q] \in C_+(I) \times C(I)\} \tag{4}$$

is the set of all such operators. Finally, for an arbitrary $r \in \mathbb{R}$ the notation $\chi_r : I \to \mathbb{R}$ stands for the constant function with value r.

Proposition 1 of [7] states that if we define multiplication of operators by

$$L(p_1, q_1) \cdot L(p_2, q_2) = L(p_1 p_2, p_1 q_2 + q_1) \tag{5}$$

and if we define that $L(p_1, q_1) \leq L(p_2, q_2)$ if

$$p_1(x) = p_2(x),\ q_1(x) \leq q_2(x) \text{ for any } x \in I, \tag{6}$$

then $(\mathbb{LA}_2(I), \cdot, \leq)$ is a noncommutative partially ordered group with the unit element (identity) $L(\chi_1, \chi_0)$. Using Lemma 3.1, Lemma 3.2 and Lemma 3.4 we get that if we put

$$\begin{aligned} &L(p_1, q_1) * L(p_2, q_2) = \\ &= \{L(p, q) \in \mathbb{LA}_2(I); L(p_1, q_1) \cdot L(p_2, q_2) \leq L(p, q)\} = \\ &= \{L(p_1 p_2, q); q \in C(I), p_1 q_2 + q_1 \leq q\} \end{aligned}, \tag{7}$$

then $(\mathbb{LA}_2(I), *)$ is a transposition hypergroup ([7], Theorem 3).

In Example 3.8 we can see that unlike in Examples 3.6 or 3.7, the elements of H (in case of Example 3.8, H is $\mathbb{LA}_2(I)$) have two components.

Furthermore, Křehlík [15, 16] has been interested in the issue of structured systems and multiautomata in analysis of processes and signals. In this respect one may study and make use of phenomena such as Gaussian–shaped pulsed signals or Chapman-Richard's models of growth (for an application of this model see [1]).

Example 3.9. Consider the function of the Gaussian-shaped pulse signal $v(t) = a\exp(-2\pi t^2)$, where $a \in \mathbb{R}^+$. When regarding the second order linear differential equation in the Jacobi form, i.e. $v''(t) + p(t)v(t) = 0$, where p is a continuous function, and creating hyperstructures of the respective linear differential operators using the *Ends lemma* following the pattern of Example 3.8, we see that we get a one-parametric system, i.e. an analogy of the simple Examples 3.6 or 3.7, as the operators have the form $L(0, \varphi(a))$, where φ stands for a suitable function of $a \in \mathbb{R}^+$.

However, if we want to apply similar reasoning on e.g. Chapman-Richard's function $y(t) = A \cdot [1 - exp(-ct)]^b$, we see that in the same context the linear differential operators are of the form $L(0, \varphi(b, c))$. Notice that not only two-parametric but also three-parametric Chapman-Richard's models are used, which would result in the linear differential operators of the form $L(0, \varphi)$, where φ is a function of three variables.

Roughly speaking, in order to construct EL–hyperstructures the first thing we need is a single-valued quasi-ordered semigroup. *Yet how can we reasonably and meaningfully define single-valued operations for more-component elements (or for arity higher than* 2*) so that we get single-valued semigroups with reasonable and meaningful quasi-orderings?*

4. **Binary context, more-component elements**

When giving an answer to the above stated question, one might use an *ad hoc approach*, i.e. treat each case as sui generis and base the definition of the single valued operation on specific properties of the objects studied. This is e.g. the case of the set of operators studied in Chvalina – Křehlík – Novák presentation given at the 12th AHA concerning the Volterra operators and the Laplace transform.

However, in this paper we are going to use the *systematic approach*, i.e. test the most likely and common operations which can be performed on an arbitrary set. We are going to demonstrate this approach on two–parametric systems, i.e. all $a \in H$, where H is the set which will be used as the basis of our considerations in Lemma 3.1, will be of the form $a = (a_1, a_2)$, where components a_1, a_2 are of a suitable type (number, matrix, polynomial, function, etc.)

It is easy to transfer our reasoning to n–parametric systems for most cases. Notice that what we present further on is a selection of possibilities only, as naturally there exists an infinite number of possible operations which can be defined instead. Should any special properties of H or of the operation defined on it or components of its elements be required, they will always be mentioned at respective places further on in the text.

4.1. **Operations applied on all components.**

Definition 4.1. For all $a = (a_1, a_2), b = (b_1, b_2) \in H$, where H is a suitable set, define $\cdot : H \times H \to H$ by

$$a \cdot b = (a_1 + a_2 + b_1 + b_2, a_1 \oplus a_2 \oplus b_1 \oplus b_2),$$

where $+, \oplus$ are suitable operations applied on components of elements of H.

Theorem 4.2. *Let H be a set of elements of the form $a = (a_1, a_2)$. Define an operation $\cdot$ on H using Definition 4.1. Then $(H, \cdot)$ is a semigroup if operations $+, \oplus$ are identical and simultanously they are associative, commutative and idempotent.*

Proof. Suppose an arbitrary triplet of elements $a, b, c \in H$ such that $a = (a_1, a_2), b = (b_1, b_2), c = (c_1, c_2)$. We need to prove that $a \cdot (b \cdot c) = (a \cdot b) \cdot c$. As far as the left-hand side is concerned, we get that

$$b \cdot c = (b_1 + b_2 + c_1 + c_2, b_1 \oplus b_2 \oplus c_1 \oplus c_2)$$

and

$$a \cdot (b \cdot c) = (a_1 + a_2 + (b_1 + b_2 + c_1 + c_2) + (b_1 \oplus b_2 \oplus c_1 \oplus c_2), a_1 \oplus a_2 \oplus (b_1 + b_2 + c_1 + c_2) \oplus (b_1 \oplus b_2 \oplus c_1 \oplus c_2)).$$

As far as the right-hand side is concerned,

$$a \cdot b = (a_1 + a_2 + b_1 + b_2, a_1 \oplus a_2 \oplus b_1 \oplus b_2)$$

and

$$(a \cdot b) \cdot c = ((a_1 + a_2 + b_1 + b_2) + (a_1 \oplus a_2 \oplus b_1 \oplus b_2) + c_1 + c_2, (a_1 + a_2 + b_1 + b_2) \oplus (a_1 \oplus a_2 \oplus b_1 \oplus b_2) \oplus c_1 \oplus c_2).$$

One can easily see that if the operations $+$ and $\oplus$ are identical and on top of that if they are associative, commutative and idempotent, both sides of the equality reduce to $(a_1 + a_2 + b_1 + b_2 + c_1 + c_2, a_1 + a_2 + b_1 + b_2 + c_1 + c_2)$. □

Example 4.3. Let $H = \{(P, R); P, R \subseteq \mathcal{P}(S)\}$, where S is a suitable set. For $(A, B), (C, D) \in H$ define

$$(A, B) \cdot (C, D) = (A \cap B \cap C \cap D, A \cap B \cap C \cap D).$$

Then $(H, \cdot)$ is a semigroup, which (after we supplement it with a suitable quasi-ordering $\leq$) may be taken as a basis for constructing the EL–semihypergroup by

$$(A, B) * (C, D) = [(A, B) \cdot (C, D))_{\leq}$$

using Lemma 3.1.

4.2. **Component-wise operations.**

Definition 4.4. For all $a = (a_1, a_2), b = (b_1, b_2) \in H$, where H is a suitable set, define $\cdot : H \times H \to H$ by

$$a \cdot b = (a_1 + b_1, a_2 \oplus b_2),$$

where $+, \oplus$ are suitable operations applied on components of elements of H.

Theorem 4.5. *Let H be a set of elements of the form $a = (a_1, a_2)$. Define an operation $\cdot$ on H using Definition 4.4. Then $(H, \cdot)$ is a semigroup if and only if both operations $+, \oplus$ are associative.*

Proof. The structure of the proof will be the same as the structure of the proof of Theorem 4.1. We get that

$$(a \cdot b) \cdot c = ((a_1 + b_1) + c_1, (a_2 \oplus b_2) \oplus c_2)$$

while

$$a \cdot (b \cdot c) = (a_1 + (b_1 + c_1), a_2 \oplus (b_2 \oplus c_2)).$$

Obviously these are equal if and only if both operations $+$ and $\oplus$ are associative. $\square$

Example 4.6. Let $H = \{(r, s); r, s \in \mathbb{R}\}$. For $(x_1, x_2), (y_1, y_2) \in H$ define

$$(x_1, x_2) \cdot (y_1, y_2) = (x_1 + y_1, x_2 y_2),$$

where the operations applied on components are the usual addition and multiplication of real numbers. Then $(H, \cdot)$ is a semigroup, which (after we supplement it with a suitable quasi-ordering $\leq$) may be taken as a basis for constructing the EL–semihypergroup by

$$(x_1, x_2) * (y_1, y_2) = [(x_1, x_2) \cdot (y_1, y_2))_{\leq}$$

using Lemma 3.1.

4.3. **Generalization of the former case.**

Definition 4.7. For all $a = (a_1, a_2), b = (b_1, b_2) \in H$, where H is a suitable set, define $\cdot : H \times H \to H$ by

$$a \cdot b = (k_1(a_1) + k_2(b_1), l_1(a_2) \oplus l_2(b_2)),$$

where $+, \oplus$ are suitable operations applied on components of elements of H and k_i, l_i, $i = 1, 2$, are suitable functions.

In the above mentioned definition we are interested in cases such as the one mentioned in the following example.

Example 4.8. Let $H = \{(r, s); r, s \in \mathbb{R}\}$. For $(a_1, a_2), (b_1, b_2) \in H$ define

$$(a_1, a_2) \cdot (b_1, b_2) = (\sin a_1 + \mathrm{e}^{b_1}, |a_2| \ln b_2).$$

If we apply the same reasoning as suggested in the proofs of Theorem 4.2 and Theorem 4.5, we get that

$$a \cdot b = (k_1(a_1) + k_2(b_1), l_1(a_2) \oplus l_2(b_2))$$

and

$$(a \cdot b) \cdot c = (k_1(k_1(a_1) + k_2(b_1)) + k_2(c_1), l_1(l_1(a_2) \oplus l_2(b_2)) \oplus l_2(c_2)) \tag{8}$$

while

$$b \cdot c = (k_1(b_1) + k_2(c_1), l_1(b_2) \oplus l_2(c_2))$$

and

$$a \cdot (b \cdot c) = (k_1(a_1) + k_2(k_1(b_1) + k_2(c_1)), l_1(a_2) \oplus l_2(l_1(b_2) \oplus l_2(c_2))). \tag{9}$$

And it is obvious that the equality $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ does not hold in general. Yet if $k_1 = k_2$ (denote these by k), $l_1 = l_2$ (denote these by l), then $\cdot$ is associative if and only if

$$k(k(a_i) + k(b_i)) + k(c_i) = k(a_i) + k(k(b_i) + k(c_i)), \text{ for } i = 1, 2 \tag{10}$$

for all $(a_1, a_2), (b_1, b_2), (c_1, c_2) \in H$ (and likewise for l). Within this we may identify some special cases where the condition (10) holds. These include e.g. cases when

(1) $k = l = Id$, i.e. the case introduced by Definition 4.4,
(2) k, l are constant mappings and $+$ and $\oplus$ are associative,
(3) k, l are homomorphisms and $k(k(x)) = k(x)$ for all $x \in \mathrm{Dom(k)}$ (and likewise for l).

Example 4.9. Let $H = \{(r, s); r, s \in \mathbb{R}\}$. For $(a_1, a_2), (b_1, b_2) \in H$ define

$$(a_1, a_2) \cdot (b_1, b_2) = (|a_1| + |b_1|, |a_2| + |b_2|).$$

Then $(H, \cdot)$ is a semigroup, which (after we supplement it with a suitable quasi ordering $\leq$) may be taken as a basis for constructing the EL–semihypergroup by

$$(a_1, a_2) * (b_1, b_2) = [(a_1, a_2) \cdot (b_1, b_2))_{\leq}.$$

using Lemma 3.1.

Remark 4.10. Notice that in Example 4.9 condition (10) holds, yet the example is of none of the special types discussed above. Also, the components need not make use of the same operation. One of them might e.g. be multiplication of absolute values instead of their sum.

4.4. **A special case of numerical k_i, l_i, $i = 1, 2$.** Suppose now that instead of regarding arbitrary functions k_i, l_i, $i = 1, 2$ we regard multiplication by a constant technically performed in accordance with the nature of components of the elements of H. In other words, for numerical components we regard multiplication of the component by a fixed number, for components being matrices we regard multiples of matrices, for polynomials multiples of polynomials, for functions pointwise multiplication by a constant, etc. Thus instead of $k_i(a_j)$, $l_i(a_j)$, $i, j = 1, 2$ we will write $k_i a_j, l_i a_j$.

Example 4.11. Let $H = \{(r, s); r, s \in \mathbb{R}\}$. For $(a_1, a_2), (b_1, b_2) \in H$ define

$$(a_1, a_2) \cdot (b_1, b_2) = (2a_1 + 3b_1, \frac{1}{2}a_2 \frac{3}{4}b_2),$$

i.e. in this particular case there is $k_1 = 2$, $k_2 = 3$, $l_1 = \frac{1}{2}$, $l_2 = \frac{3}{4}$.

Definition 4.12. For all $a = (a_1, a_2), b = (b_1, b_2) \in H$, where H is a suitable set, define $\cdot : H \times H \to H$ by

$$a \cdot b = (k_1 a_1 + k_2 b_1, l_1 a_2 \oplus l_2 b_2),$$

where $+, \oplus$ are suitable operations applied on components of elements of H and k_i, l_i, $i = 1, 2$, are fixed real numbers.

Theorem 4.13. *Let H be a set of elements of the form $a = (a_1, a_2)$. Define an operation $\cdot$ on H using Definition 4.12. Then $(H, \cdot)$ is a semigroup if and only if the multiplication by k_i, for $i = 1, 2$, is distributive over $+$ (and likewise multiplication by l_i for $i = 1, 2$ distributive over $\oplus$), there is $k_i(k_j a_m) = (k_i k_j) a_m = k_i k_j a_m$ for $i, j, m \in \{1, 2\}$ (and likewise for l) and simultaneously $k_i, l_i \in \{0, 1\}$ for $i = 1, 2$.*

Proof. In this new context the reasoning included in Subsection 4.3 changes (under the condition of distributivity) to

$$(a \cdot b) \cdot c = (k_1 k_1 a_1 + k_1 k_2 b_1 + k_2 c_1, l_1 l_1 a_2 + l_1 l_2 b_2 + l_2 c_2)$$

while
$$a\cdot(b\cdot c)=(k_1a_1+k_2k_1b_1+k_2k_2c_1,l_1a_2+l_2l_1b_2+l_2l_2c_2).$$
Since we must secure that multiplication of coefficients k_i, l_i, $i=1,2$ is idempotent, the choice of their values is obvious. $\square$

Remark 4.14. In fact we require multiplication of coefficients k, l to have some properties of an outer operation on $(H,\cdot)$.

Example 4.15. Let $H=\{(r,s);r,s\in\mathbb{R}\}$. For $(a_1,a_2),(b_1,b_2)\in H$ define
$$(a_1,a_2)\cdot(b_1,b_2)=(a_1,\min\{a_2,b_2\})$$
Then $(H,\cdot)$ is a semigroup which can be taken as a basis for an EL–semihypergroup.

Example 4.16. Let $H=\{(P,R);P,R\subseteq\mathcal{P}(S)\}$, where S is a suitable set. For $(A,B),(C,D)\in H$ define
$$(A,B)\cdot(C,D)=(A\cup C,D).$$
Then $(H,\cdot)$ is a semigroup which can again be taken as a basis for an EL–semihypergroup..

4.5. **Functions applied on components.** We may point out another special case of Definition 4.7 – application of a given function on certain specified components of elements of H. By setting $k_1=f,k_2\equiv 0$, $l_1\equiv 0,l_2=g$ we get the following.

Definition 4.17. For all $a=(a_1,a_2),b=(b_1,b_2)\in H$, where H is a suitable set, define $\cdot:H\times H\to H$ by
$$a\cdot b=(f(a_1),g(b_2)),$$
where f,g are suitable functions applied on the components.

Corollary 4.18. *Let H be a set of elements of the form $a=(a_1,a_2)$. Define an operation $\cdot$ on H using Definition 4.17. Then $(H,\cdot)$ is a semigroup if and only if $f(x)=f(f(x))$ and $g(y)=g(g(y))$ for all $x\in Dom(f)$ or $y\in Dom(g)$ respectively.*

Proof. Follows immediately from (8) and (9) by setting $k_1=f,k_2\equiv 0$, $l_1\equiv 0,l_2=g$. $\square$

Remark 4.19. Cases such as $a\cdot b=(f(a_2),g(b_1))$ are associative only in very special cases (such as $f\equiv g$ and both being constant).

Example 4.20. Let $H_{\mathbb{R}}=\{(r,s);r,s\in\mathbb{R}\}$. For $(a_1,a_2),(b_1,b_2)\in H$ define
$$(a_1,a_2)\cdot(b_1,b_2)=(|a_1|,\operatorname{sgn}(b_2)).$$
Then $(H_{\mathbb{R}},\cdot)$ is a semigroup.

Example 4.21. Let $H_M=\{(M_1,M_2);M_1,M_2\in\mathrm{SqMat}\}$, where SqMat is the set of all square matrices (regardless of size). For $(M_1,M_2),(N_1,N_2)\in H$ define
$$(M_1,M_2)\cdot(N_1,N_2)=(\det(M_1),\operatorname{tr}(N_2)).$$
Then $(H_M,\cdot)$ is a semigroup.

In Example 4.21 we have used determinant and trace of a matrix. In the context of polynomials we could use instruments such as the sum of its coefficients, coefficient of the absolute term, derivative of a suitable order, etc.

4.6. **Comparison of the two approaches.** Compare Example 3.8, where

$$L(p_1, q_1) \cdot L(p_2, q_2) = L(p_1 p_2, p_1 q_2 + q_1). \tag{11}$$

and the above Example 4.21, where

$$(M_1, M_2) \cdot (N_1, N_2) = (\det(M_1), \mathrm{tr}(N_2)). \tag{12}$$

One can see that the result of the multiplication is either "of the same type" (a fully meaningful operator as in (11)) or "of different quality" (a matrix which is however a matrix only formally, because its potential lies in the fact that it can be treated as a number). In the following example, which is a continuation of Example 4.21, we show that the latter approach may be well used in the context of Lemma 3.1 and EL–hyperstructures.

Example 4.22. For $(M_1, M_2), (N_1, N_2) \in H_M$ define

$$(M_1, M_2) \leq (N_1, N_2) \Leftrightarrow \det(M_1) = \det(N_1) \text{ and } \mathrm{tr}(M_2) \leq \mathrm{tr}(N_2)$$

and define

$$(M_1, M_2) * (N_1, N_2) = [(M_1, M_2) \cdot (N_1, N_2))_{\leq}.$$

Then based on Lemma 3.1 we get that $(H_M, *)$ is a semihypergroup.

For more ideas for (and obstacles when) defining EL–hyperstructures on sets of matrices cf. e.g. [22].

4.7. **Some negative examples.** Examples of operations which in spite of being "logical picks" are not associative include definitions of $\cdot$ such that e.g.:

(1) $a \cdot b = (f(a_1 + a_2), g(b_1 \oplus b_2))$, where f, g are functions; not even for $f \equiv g$, $+ \equiv \oplus$,
(2) $a \cdot b = (f(a_1 + b_1), g(a_2 \oplus b_2))$, i.e. functions applied on the result of component-wise operations; if f, g are not homomorphisms, then this is associative only in very special contexts,
(3) "combining components" such as e.g. $a \cdot b = (b_2, a_1)$ or $a \cdot b = (f(a_1), g(b_1))$.

5. n–ary context

From the binary context of Lemma 3.1, where the hyperoperation on a semigroup $(H, \cdot)$ is defined by

$$a * b = [a \cdot b)_{\leq} = \{x \in H; a \cdot b \leq x\}$$

we may proceed to the n–ary context, i.e. define

$$\underbrace{a_1 * \ldots * a_n}_{n} = [\underbrace{a_1 \cdot \ldots \cdot a_n}_{n})_{\leq} = \{x \in H; \underbrace{a_1 \cdot \ldots \cdot a_n}_{n} \leq x\}$$

or in the standard n–ary notation

$$f(a_1^n) = [\underbrace{a_1 \cdot \ldots \cdot a_n}_{n})_{\leq} = \{x \in H; \underbrace{a_1 \cdot \ldots \cdot a_n}_{n} \leq x\}.$$

Yet prior to doing this two important issues must be considered:

(1) How do we obtain the single-valued product, i.e. what is the arity of the single-valued operation?
(2) Is $*$ (i.e. f) an n–ary or an iterated binary hyperoperation?

The n–ary context of the *Ends lemma* was first examined in an example included in [11], which was a generalisation of an example included in [14]. However, it was properly studied in detail only in [20], where the following two results are included. Both of them are based on the idea that in $f(a_1^n) = [\underbrace{a_1 \cdot \ldots \cdot a_n}_{n})_\le = \{x \in H; \underbrace{a_1 \cdot \ldots \cdot a_n}_{n} \le x\}$ we consider the iterated binary single-valued operation and an n–ary hyperoperation.

Theorem 5.1. [20] *Let $(H, \cdot, \le)$ be a quasi-ordered semigroup. n–ary hyperoperation $f : H^n \to \mathcal{P}^*(H)$ defined as*

$$f(a_1^n) = [\underbrace{a_1 \cdot \ldots \cdot a_n}_{n})_\le = \{x \in H; \underbrace{a_1 \cdot \ldots \cdot a_n}_{n} \le x\}.$$

is associative. Furthermore, it is commutative if the semigroup $(H, \cdot)$ is commutative.

Theorem 5.2. [20] *Let $(H, \cdot, \le)$ be a quasi-ordered group. n–ary EL–semihypergroup constructed as above is an n–ary hypergroup.*

For details and further results concerning the n–ary context of the construction cf. [20].

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