ON A SEQUENCE OF FINITE H_v-GROUPS

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ABSTRACT. Defining a hyperoperation on a finite set H, which its elements consist of one special element together with n elements with indices belonging to $\mathbb{Z}_n$, a single-power cyclic H_v-group results. This paper deals not only with the study of this H_v-group and its properties but also the study of a sequence of H_v-structures depending on the values of n of $\mathbb{Z}_n$, together with topological examples on $\mathbb{R}^2$ introducing the concept of the *boundary* as a kind of hyperoperation identical to the above.

1. Introduction

An algebraic hyperstructure is a non empty set H endowed with at least one hyperoperation i.e. a multivalued operation, that associates with two elements of H not an element, as in a classical structure, but a subset of H. Therefore, if in a set H at least one hyperstructure $\cdot$:H$\times$ H$\to$ P(H) $-$ $\{\emptyset\}$ is defined, then (H,$\cdot$) is called a hypergroupoid. The H_v-structures introduced in 1990 [7], is the largest class of hyperstructures. The H_v-structures satisfy the weak axioms where the non-empty intersection replaces the equality, as below:

$$\text{weak associativity: } (xy)z \cap x(yz) \neq \emptyset, \forall x, y, z \in \text{H}$$
$$\text{weak commutativity: } xy \cap yx \neq \emptyset, \forall x, y \in \text{H}$$

The hyperstructure (H,$\cdot$) is called H_v-semigroup if it is weak associative and it is called H_v-group if it is reproductive H_v-semigroup, i.e. xH=Hx=H, $\forall x \in$ H. If in a H_v-group the weak commutativity is valid then it is called H_v-commutative group.
If there exists $h \in$ H and $s \in \mathbb{Z}^+$, the minimum one, such that H= h^s then H will be called single-power cyclic with generator h and period s.
A H_v-group is called H_b-group if its hyperoperation contain operation which define a group.

Recall some basic definitions [8], [2], [3]:
Let (H,$\cdot$) be a hypergroupoid. An element $e \in H$ is called scalar if xe = ex =

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singleton, $\forall x \in H$ and it is called scalar unit element if xe = ex = x, $\forall x \in H$. An element $x' \in H$ is called an inverse of $x \in H$ if there exists a unit $e \in H$, such that $e \in x \cdot x' \cap x' \cdot x$. Let us denote by $I^l_\cdot(x, e)$ [4] the set of the left inverses of the element $x \in H$ associated with the left unit $e \in H$ with respect to hyperoperation $(\cdot)$, (resp. $I^r_\cdot(x, e)$ is the set of the right inverses of the element $x \in H$ associated with the right unit $e \in H$ with respect to hyperoperation $(\cdot)$). An element $a \in H$ is called idempotent element if $a^2 = a$. The n^{th} power of an element h, denoted h^n, $n > 1$, is defined to be the union of all expressions of n times of h, in which the parentheses are put in all possible ways.

The present paper deals with a sequence of H_v-structures. Firstly, we define a hyperoperation on a set and study, in the general case, the hyperstructure resulting. The hyperstructure is a H_v-group. The hyperoperation is defined in every finite hyperstructure using indices of the cyclic group $\mathbb{Z}_n$. The case of infinite order is separately studied with indices in $\mathbb{N}$. We also study the existence of identities, inverses elements and powers of the elements of the H_v-group. The hyperstructures of small order, provided with the particular hyperoperation are groups, those of greater order are hypergroups and then those of the greatest order lead to H_v-groups. A sequence of finite H_v-groups with common properties is created. Finally, we present the motivating example. The hyperoperation is defined in a geometrical figure on $\mathbb{R}^2$ [1], [4], which is partitioned into a finite or infinite number of parts. The hyperoperation is defined in the sense of the *boundary* among the parts. A lot of properties of this geometrical H_v-structures are studied.

2. **The general case**

Consider the set H = $\{\alpha\} \cup \{z \mid z = a_r, r \in \mathbb{Z}_n\}$.
On H let us define the hyperoperation $(\cdot)$ as follows:

Definition 2.1. For every $x, y \in H$ define

$$(\cdot):\ \text{H}\times\ \text{H} \rightarrow P(\text{H})\text{-}\{\emptyset\} : (x, y) \mapsto x \cdot y$$

such that

$$x \cdot y = \begin{cases} \{\alpha, a_{p-1}, a_{p+1}, a_{k-1}, a_{k+1}\}, x = a_p, y = a_k, \forall p, k \in \mathbb{Z}_n \\ x \cdot \alpha = \alpha \cdot x = x, \forall x \in H \end{cases}$$

Some properties of the hyperoperation $(\cdot)$:
1. Obviously $(\cdot)$ is commutative, i.e $x \cdot y = y \cdot x, \forall x, y \in H$.
2. According to Definition 2.1, $\alpha \cdot \alpha = \alpha$ and $\alpha \cdot a_p = a_p \cdot \alpha = a_p$, $\forall p \in \mathbb{Z}_n$. That means that $\alpha \cdot x = x \cdot \alpha = x$,$\forall x \in H$, so the element α is scalar unit element.
3. $\alpha^m = \alpha$, $m \in \mathbb{N}^*$.
4. Since $\alpha^2 = \alpha$, the element α is also an idempotent element and obviously $I_\cdot(\alpha, \alpha) = \alpha$.
5. $|H| = n + 1$

6. $H/\beta^* = H$
7. $x \cdot y = x^2 \cup y^2$, $\forall x, y \in H$ with $(x, y) \neq (\alpha, y)$ and $(x, y) \neq (x, \alpha)$ [6]

Proposition 2.2. *$I.(a_p, \alpha) = H - \{\alpha\}$, $\forall p \in \mathbb{Z}_n$*

Proof. Let $a_k \in I.(a_p, \alpha)$ where $k, p \in \mathbb{Z}_n \Rightarrow \alpha \in a_k \cdot a_p$ and $\alpha \in a_p \cdot a_k \Rightarrow \alpha \in \{\alpha, a_{k-1}, a_{k+1}, a_{p-1}, a_{p+1}\}$ and $\alpha \in \{\alpha, a_{p-1}, a_{p+1}, a_{k-1}, a_{k+1}\}$, since the previous relations are both true for every $k, p \in \mathbb{Z}_n$, we get that $I.(a_p, \alpha) = H - \{\alpha\}$, $\forall p \in \mathbb{Z}_n$.□

Proposition 2.3. $\displaystyle\bigcup_{x \in H} x^2 = \bigcup_{k \in \mathbb{Z}_n} a_k^2 = H$

Proof. Notice that

$$\bigcup_{k \in \mathbb{Z}_n} a_k^2 = \bigcup_{k \in \mathbb{Z}_n} \{\alpha, a_{k-1}, a_{k+1}\} = \{\alpha, a_{-1}, a_1, a_0, a_2, a_3, \ldots, a_{n-1}, a_n\} =$$
$$= \{\alpha, a_0, a_1, a_2, a_3, \ldots, a_{n-1}\} = H, \text{ since } a_{-1} = a_{n-1} \text{ and } a_n = a_0.$$

Also, since $\alpha^2 = \alpha$, we get that $\displaystyle\bigcup_{x \in H} x^2 = \bigcup_{k \in \mathbb{Z}_n} a_k^2 = H$.□

Proposition 2.4. $a_k^2 \subseteqq a_p \cdot a_k$, $\forall k, p \in \mathbb{Z}_n$

Proof. For $k, p \in \mathbb{Z}_n$

$$a_p \cdot a_k = \{\alpha, a_{p-1}, a_{p+1}, a_{k-1}, a_{k+1}\} \supset \{\alpha, a_{k-1}, a_{k+1}\} = a_k^2$$

When $p = k$ we get that $a_k \cdot a_k = \{\alpha, a_{k-1}, a_{k+1}\} = a_k^2$.
So, generally, $a_k^2 \subseteqq a_p \cdot a_k$, $\forall k, p \in \mathbb{Z}_n$.□

Proposition 2.5. *The hyperstructure $(H, \cdot)$ is a commutative H_v-group.*

Proof. For the reproduction axiom, let $x = \alpha$, then

$$x \cdot H = \alpha \cdot H = \bigcup_{h \in H} (\alpha \cdot h) = \bigcup_{h \in H} h = H = H \cdot \alpha = H \cdot x$$

Let $x = \alpha_k, k \in \mathbb{Z}_n$, then

$$x \cdot H = \alpha_k \cdot H = \bigcup_{h \in H} (\alpha_k \cdot h) = (\alpha_k \cdot \alpha) \cup [\bigcup_{k \in \mathbb{Z}_n} (\alpha_k \cdot \alpha_p)] =$$
$$= \{a_k\} \cup \{\alpha, a_{k-1}, a_{k+1}, a_{-1}, a_0, a_1, a_2, a_3, \ldots, a_{k-1}, a_{k+1}, \ldots, a_{n-2}, a_n\} =$$
$$= \{\alpha, a_0, a_1, a_2, a_3, \ldots, a_{n-2}, a_{n-1}\} = H, \text{ since } a_{-1} = a_{n-1} \text{ and } a_n = a_0.$$

Obviously $H \cdot a_k = H \cdot x = H$, then

$$x \cdot H = H \cdot x = H \ , \forall x \in H.$$

Since $(\cdot)$ is commutative, according to [5], we have to check only the following cases for the associativity: For $p, k \in \mathbb{Z}_n$
• $\alpha \cdot (\alpha \cdot a_k) = \alpha \cdot a_k = a_k$ and $(\alpha \cdot \alpha) \cdot a_k = \alpha \cdot a_k = a_k$
• $\alpha \cdot (a_p \cdot a_k) = \{\alpha, a_{p-1}, a_{p+1}, a_{k-1}, a_{k+1}\} = a_p \cdot a_k$ and $(\alpha \cdot a_p) \cdot a_k = a_p \cdot a_k$
• $a_k \cdot (a_k \cdot \alpha) = a_k \cdot a_k = \{\alpha, a_{k-1}, a_{k+1}\}$ and $(a_k \cdot a_k) \cdot \alpha = \{\alpha, a_{k-1}, a_{k+1}\}$
So far, notice that if one or two elements α appear in the triples (x, y, z), then the equality appears for the associativity, i.e $x \cdot (y \cdot z) = (x \cdot y) \cdot z$.
Furthermore, we have to check the following two cases: For $p, k, m \in \mathbb{Z}_n$
• $a_k \cdot (a_k \cdot a_p) = a_k \cdot \{\alpha, a_{k-1}, a_{k+1}, a_{p-1}, a_{p+1}\} =$
$= \{a_k, a_{k-1}, a_{k+1}, a_{k-2}, a_{k+2}, a_p, a_{p-2}, a_{p+2}\}$
and $(a_k \cdot a_k) \cdot a_p = \{\alpha, a_{k-1}, a_{k+1}\} \cdot a_p = \{a_p, a_{k-2}, a_k, a_{p-1}, a_{p+1}, a_{k+2}\}$
So,

$$a_k \cdot (a_k \cdot a_p) \cap (a_k \cdot a_k) \cdot a_p = \{a_p, a_k, a_{k-2}, a_{k+2}\} \neq \emptyset.$$

• $a_k \cdot (a_p \cdot a_m) = \{a_k, a_{k-1}, a_{k+}, a_p, a_{p-2}, a_{p+2}, a_m, a_{m-2}, a_{m+2}\}$
and $(a_k \cdot a_p) \cdot a_m = \{a_m, a_{k-2}, a_k, a_{m-1}, a_{m+1}, a_{k+2}, a_p, a_{p-2}, a_{p+2}\}$.
So,

$$a_k \cdot (a_p \cdot a_m) \cap (a_k \cdot a_p) \cdot a_m = \{a_p, a_k, a_m, a_{p-2}, a_{p+2}\} \neq \emptyset.$$

So, generally

$$x \cdot (y \cdot z) \cap (x \cdot y) \cdot z \neq \emptyset, \forall x, y \in H.\square$$

Proposition 2.6. $a_k^p \subseteq a_k^m, \forall k \in \mathbb{Z}_n, p, m \in \mathbb{N}, p, m > 1 \Leftrightarrow m \geqq p$

Proof. Since $(\cdot)$ is commutative, for $k \in \mathbb{Z}_n$:
$a_k^2 = \{\alpha, a_{k-1}, a_{k+1}\}$
$a_k^3 = a_k^2 \cdot a_k = \{\alpha, a_{k-2}, a_{k-1}, a_k, a_{k+1}, a_{k+2}\}$
$a_k^4 = a_k^3 \cdot a_k \cup a_k^2 \cdot a_k^2 =$
$= \{\alpha, a_{k-3}, a_{k-2}, a_{k-1}, a_k, a_{k+1}, a_{k+2}, a_{k+3}\} \cup \{\alpha, a_{k-2}, a_{k-1}, a_k, a_{k+1}, a_{k+2}\} =$
$= \{\alpha, a_{k-3}, a_{k-2}, a_{k-1}, a_k, a_{k+1}, a_{k+2}, a_{k+3}\}$.
So, by induction, for $p, m \in \mathbb{N}, p, m > 1$ we get:

$$a_k^p = \{\alpha, a_{k-p+1}, a_{k-p+2}, \ldots, a_{k+p-2}, a_{k+p-1}\}$$
$$a_k^m = \{\alpha, a_{k-m+1}, a_{k-m+2}, \ldots, a_{k+m-2}, a_{k+m-1}\}.$$

Let $a_k^p \subseteq a_k^m \Rightarrow k - m + 1 \leqq k - p + 1$ and $k + m - 1 \geqq k + p - 1 \Rightarrow m \geqq p$.
Now, let $m \geqq p \Rightarrow k - m + 1 \leqq k - p + 1$ and $k + m - 1 \geqq k + p - 1, k \in \mathbb{Z}_n$.
Since $k - p + 1, k + p - 1, k - m + 1, k + m - 1 \in \mathbb{Z}_n \Rightarrow$
$\Rightarrow \{a_{k-p+1}, a_{k-p+2}, \ldots, a_{k+p-1}\} \subseteq \{a_{k-m+1}, a_{k-m+2}, \ldots, a_{k+m-1}\} \Rightarrow$
$\Rightarrow \{\alpha, a_{k-p+1}, a_{k-p+2}, \ldots, a_{k+p-1}\} \subseteq \{\alpha, a_{k-m+1}, a_{k-m+2}, \ldots, a_{k+m-1}\} \Rightarrow$
$\Rightarrow a_k^p \subseteq a_k^m.\square$

Corollary 2.7. $a_k^m = a_k^{m-1} \cdot a_k, \forall k \in \mathbb{Z}_n, m \in \mathbb{N}, m > 1$

Proof. For $k \in \mathbb{Z}_n, m \in \mathbb{N}, m > 1$, since $(\cdot)$ is commutative:
$a_k^m = a_k^{m-1} \cdot a_k \cup a_k^{m-2} \cdot a_k^2 \cup \ldots \cup a_k^{m/2} \cdot a_k^{m/2}$, if $m = 2p, p \in \mathbb{N}$

$a_k^m = a_k^{m-1} \cdot a_k \cup a_k^{m-2} \cdot a_k^2 \cup \ldots \cup a_k^{m+1/2} \cdot a_k^{m-1/2}$, if $m = 2p+1, p \in \mathbb{N}$
According to Proposition 2.6, the power $m-1$ is the greatest one in both cases, so:

$$a_k^m = a_k^{m-1} \cdot a_k, \forall k \in \mathbb{Z}_n, m \in \mathbb{N}, m > 1. \square$$

Proposition 2.8. *The hyperstructure $(H, \cdot)$ is a single-power cyclic H_v-group and each element $a_k \in H, k \in \mathbb{Z}_n, n = 2, 3$ is a generator with period 3.*

Proof. Let $x = a_k, k \in \mathbb{Z}_n$ be any element of H, then:
$a_k^2 = \{\alpha, a_{k-1}, a_{k+1}\}$
$a_k^3 = a_k^2 \cdot a_k = \{\alpha, a_{k-2}, a_{k-1}, a_k, a_{k+1}, a_{k+2}\}$
For $n = 2, H = \{\alpha, a_0, a_1\}$:
Notice that $|H| = 3$ and $|a_k^2| = 3$ but $a_k \notin a_k^2$, so $a_k^2 \neq H$.
Since $|a_k^3| = 6$ and $a_k \in a_k^3 \Rightarrow a_k^3 = H$.
That means that $(\{\alpha, a_0, a_1\}, \cdot)$ is single-power cyclic H_v-group and each element $a_k \in H, k \in \mathbb{Z}_2$ is generator with period 3.
For $n = 3, H = \{\alpha, a_0, a_1, a_2\}$:
Notice that $|H| = 4$, $|a_k^3| = 6$ and $a_k \in a_k^3 \Rightarrow a_k^3 = H$.
That means that $(\{\alpha, a_0, a_1, a_2\}, \cdot)$ is single-power cyclic H_v-group and each element $a_k \in H, k \in \mathbb{Z}_3$ is generator with period 3. $\square$

Proposition 2.9. *The hyperstructure $(H, \cdot)$ is a single-power cyclic H_v-group and each element $a_k \in H, k \in \mathbb{Z}_n$ is a generator with period the minimum $m \in \mathbb{N}, m > 1$ such that $m \geqq \frac{n+1}{2}, n \geqq 4$.*

Proof. Let $x = a_k, k \in \mathbb{Z}_n, n \geqq 4$ be any element of H, then from Proposition 2.6:

$$a_k^m = \{\alpha, a_{k-m+1}, a_{k-m+2}, \ldots, a_{k+m-2}, a_{k+m-1}\}, m \in \mathbb{N}, m > 1$$

Notice that $|a_k^m| = 1 + [(k+m-1) - (k-m+1) + 1] = 2m$.
Since $|H| = n+1$, we get that

$$a_k^m = H \Rightarrow 2m \geqq n+1 \Rightarrow m \geqq \frac{n+1}{2} . \square$$

Consider now the infinite set $H' = \{\alpha\} \cup \{z / z = a_m, m \in \mathbb{N}\}$. Then $(H', \cdot)$ is also a commutative H_v-group.

Proposition 2.10. *The hyperstructure $(H', \cdot)$ is a single-power cyclic H_v-group with infinite period and each element $a_m \in H', m \in \mathbb{N}$ is a generator.*

Proof. Let $x = a_m, a_m \in H', m \in \mathbb{N}$ be any element of H', then:
$a_m^1 = a_m$

$a_m^2 = \{\alpha, a_{m-1}, a_{m+1}\}$

$a_m^3 = \{\alpha, a_{m-2}, a_{m-1}, a_m, a_{m+1}, a_{m+2}\}$

$a_m^4 = \{\alpha, a_{m-3}, a_{m-2}, a_{m-1}, a_m, a_{m+1} a_{m+2}, a_{m+3}\}$

. . . .

. . . .

$a_m^{n-1} = \{\alpha, a_{m-n+2}, a_{m-n+3}, \ldots, a_{m+n-3}, a_{m+n-2}\}, n \in \mathbb{N}^*$

$a_m^{n} = \{\alpha, a_{m-n+1}, a_{m-n+2}, \ldots, a_{m+n-2}, a_{m+n-1}\}, n \in \mathbb{N}^*$

. . . .

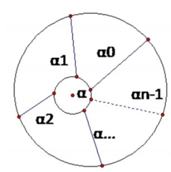
. . . .

So, every element of H' belongs to a power of a_m and there exists $n_0 \geqq 1$ such that for every $n \geqq n_0$:

$$a_m^1 \cup a_m^2 \cup a_m^3 \cup \ldots \cup a_m^{n-1} \subset a_m^n . \square$$

3. The motivating example

Let us consider the geometrical shape of the figure below. It is partitioned into $n+1$ parts such that:



(i) There is a part, denoted by α, which borders all the rest.
(ii) In addition, each of the remaining parts, denoted by $a_i, i \in \mathbb{Z}_n$ borders the two others (its adjacent ones).
Thus the set H of all parts of the figure consists of one central and n peripheral parts.

Definition 3.1. On H we introduce a hyperoperation $(*)$ such that:
(i) the hyperproduct between the central part α and any other x is equal to x, i.e. $\alpha * x = x * \alpha = x$.
(ii) the hyperproduct of two peripheral parts is the set of all parts border to them (their adjacent ones), i.e. $a_i * a_j = \{\alpha, a_{i-1}, a_{i+1}, a_{j-1}, a_{j+1}\}, i, j \in \mathbb{Z}_n$.

We call $(*)$, *boundary hyperoperation.* Notice that $* \equiv \cdot$, for example

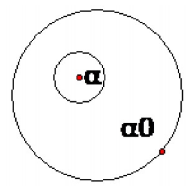
$$\alpha * \alpha = \alpha, \alpha * a_2 = a_2, a_1 * a_2 = \{\alpha, a_0, a_1, a_2, a_3\}, a_2 * a_5 = \{\alpha, a_1, a_3, a_4, a_6\}$$

We study below some cases, depending on n, of the boundary hyperoperation, $n = 1, 2, 3, 4, 5$ then a sequence of hyperstructures is created and in each one we present the corresponding figure, the Cayley table of the hyperoperation [5] and the kind of the hyperstructure resulting.

Case 1

If $n = 1$ then $H = \{\alpha, a_0\}$

$*$	α	a_0
α	α	a_0
a_0	a_0	α

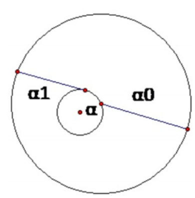


Notice that $(\{\alpha, a_0\}, *)$ is a group and $H \cong \mathbb{Z}_2$.

Case 2

If $n = 2$ then $H = \{\alpha, a_0, a_1\}$

$*$	α	a_0	a_1
α	α	a_0	a_1
a_0	a_0	α, a_1	H
a_1	a_1	H	α, a_0

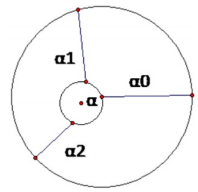


Notice that $(\{\alpha, a_0, a_1\}, *)$ is a hypergroup. It is also a H_b-group, greater than a group, isomorphic to $\mathbb{Z}_3$. We say that H contains $\mathbb{Z}_3$ up to isomorphism.

Case 3

If $n = 3$ then $H = \{\alpha, a_0, a_1, a_2\}$

$*$	α	a_0	a_1	a_2
α	α	a_0	a_1	a_2
a_0	a_0	α, a_1, a_2	H	H
a_1	a_1	H	α, a_0, a_2	H
a_2	a_2	H	H	α, a_0, a_1

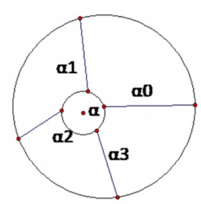


Notice that $(\{\alpha, a_0, a_1, a_2\}, *)$ is a hypergroup. It is also a H_b-group which contains $\mathbb{Z}_4$ up to isomorphism.

Case 4

If $n = 4$ then $H = \{\alpha, a_0, a_1, a_2, a_3\}$

$*$	α	a_0	a_1	a_2	a_3
α	α	a_0	a_1	a_2	a_3
a_0	a_0	α, a_1, a_3	H	α, a_1, a_3	H
a_1	a_1	H	α, a_0, a_2	H	α, a_0, a_2
a_2	a_2	α, a_1, a_3	H	α, a_1, a_3	H
a_3	a_3	H	α, a_0, a_2	H	α, a_0, a_2

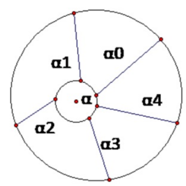


Notice that $(\{\alpha, a_0, a_1, a_2, a_3\}, *)$ is a H_v-group.

Case 5

If $n = 5$ then $H = \{\alpha, a_0, a_1, a_2, a_3, a_4\}$

$*$	α	a_0	a_1	a_2	a_3	a_4
α	α	a_0	a_1	a_2	a_3	a_4
a_0	a_0	α, a_1, a_4	α, a_0, a_1 a_2, a_4	α, a_1, a_3 a_4	α, a_1, a_2 a_4	α, a_0, a_1 a_3, a_4
a_1	a_1	α, a_0, a_1 a_2, a_4	α, a_0, a_2	α, a_0, a_1 a_2, a_3	α, a_0, a_2 a_4	α, a_0, a_2 a_3
a_2	a_2	α, a_1, a_3 a_4	α, a_0, a_1 a_2, a_3	α, a_1, a_3	α, a_1, a_2 a_3, a_4	α, a_0, a_1 a_3
a_3	a_3	α, a_1, a_2 a_4	α, a_0, a_2 a_4	α, a_1, a_2 a_3, a_4	α, a_2, a_4	α, a_0, a_2 a_3, a_4
a_4	a_4	α, a_0, a_1 a_3, a_4	α, a_0, a_2 a_3	α, a_0, a_1 a_3	α, a_0, a_2 a_3, a_4	α, a_0, a_3



Notice that $(\{\alpha, a_0, a_1, a_2, a_3, a_4\}, *)$ is a H_v-group.

References

[1] N. Antampoufis, *Hypergroups and Hb - groups in Complex Numbers*, Proceedings of 9th AHA Congress, Journal of basic Science (JBSUM), **4(1)**, Babolsar, Iran, (2008), 17-25.

[2] P. Corsini, *Prolegomena of hypergroup theory*, Aviani Editore, 1993.

[3] P. Corsini, V. Leoreanu, *Applications of Hyperstructures Theory*, Kluwer Academic Publishers, Boston/Dordrecht/London, 2003.

[4] A. Dramalidis, *Dual Hv-rings*, Rivista di Mathematica Pura ed Applicata, **17** (1996), 55-62.

[5] A. Dramalidis, N. Antampoufis, *On Hv-structures associated with binary relations*, Int. Journal of AHA (IJAHA), **1** (2014), 69-79.

[6] S. Hoskova, S. Mayerova, *Construction of subquasi - ordered hypergroup*, International Colloguim on the management of Educational Process aimed at curent issues in science, education and creative thinking development, ISBN-978-80-7231-780-6, (2011), 159-165.

[7] T. Vougiouklis, *The fundamental relation in hyperrings. The general hyperfield*, Proc. of the 4th AHA, Xanthi 1990, World Scientific,(1991) 203-211.

[8] T. Vougiouklis, *Hyperstructures and their representations*, Monographs in Mathematics, Hadronic Press, USA, 1994.