$s_2 - Hv(h/v)$-STRUCTURES AND s_2-HYPERSTRUCTURES

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To the memory of Dr. Hossein Hedayati

Abstract. The largest classes of the hyperstructures are the ones called Hv-structures, where the equality in the axioms is replaced by the non-empty intersection. The present paper deals with some new classes of hyperstructures denoted as s_2-Hv-structures and s_2-Hyperstructures. Firstly, we introduce the s_2-Hv-structures. These hyperstructures are the ones in which all their fundamental β^* classes are singleton except for two. They are, in a way, relevant to the Very Thin Hyperstructures regarding their fundamental classes. We focus our study on the notion of s_2-Hv-groups and present the kinds of them. Propositions which refer to the elements of such hyperstructures as well as to the existing types of s_2-Hv-structures are proved. Examples are developed which are relevant to the construction of such hyperstructures and their connection to S-constructions.

1. Introduction

In 1934, F. Marty [12], introduced the notion of the hypergroup. The theory of hyperstructures was subsequently developed with the contribution of various authors. Some basic definitions and theorems about hyperstructures can be found in the books [1],[5],[6],[8],[16] and the papers in the references [7],[9],[10],[15]. The main tools in hyperstructures theory are the fundamental relations. These relations, on the one hand connect the hyperstructures theory with the corresponding classical one, and on the other hand, introduce new important classes.

This paper deals with a class of hyperstructures and Hv-structures, called s_2-hyperstructures (s_2 -Hv-structures), defined using the fundamental relationship β^*. Specifically, these hyperstructures are determined by the number of fundamental β^*-classes which are not singleton. Their study is based on the behavior of the fundamental classes, particularly that of the core of the hyperstructure (Hv-structure). In the first part of chapter 4, we introduce the definition of s_2-hyperstructure, classify these hyperstructures in two types and sub-types and study some properties

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of each type. Then, criteria for the classification of hyperstructures are introduced and a construction is presented with desired fundamental structure. In the last part of this chapter, examples of s_2-hyperstructures are displayed, presenting the h/v-group one as of the most importance.

2. **Preliminaries**

In 1990, Th. Vougiouklis introduced the class of Hv-structures which satisfy the weak axioms where the non-empty intersection replaces the equality [16]. In a set H $\neq \emptyset$ equipped with a hyperoperation $(\cdot) : H \times H \to \wp^*(H)$ we abbreviate by

WASS the weak associativity: $(xy)z \cap x(yz) \neq \emptyset, \quad \forall x, y, z \in H$

COW the weak commutativity: $xy \cap yx \neq \emptyset, \quad \forall x, y \in H$.

Definition 2.1. The hyperstructure $(H, \cdot)$ is called ***Hv-semigroup*** if it is WASS and it is called ***Hv-group*** if it is reproductive Hv - semigroup, i.e. xH = Hx = H, $\forall x \in H$.

Definition 2.2. [16] The ***fundamental relation*** β^*[11] is defined in Hv-semigroups, Hv-groups, as the smallest equivalence so that the quotient would be semigroup, group respectively. A way to find the fundamental classes is given by analogous theorems[5],[6],[13],[14] to the following:

Theorem 2.3. *Let $(H, \cdot)$ be a Hv-semigroup and denote by U the set of all finite products of elements of H. We define the relation β in H by $x\beta y$ iff $\{x,y\} \subseteq u$ where u$\in U$. Then,*
the **fundamental relation** β^* **is the transitive closure of** β.

Definition 2.4. [16]An element is called ***single*** if its fundamental class is singleton. We denote S_H the set of single elements of H.

Theorem 2.5. *Let $(H, \cdot)$ be a Hv-group and $s \in S_H \neq \emptyset$. Then $sx = \beta^*(sx), \forall x \in H$. Let $(H, \cdot)$ be a Hv-group and $S_H \neq \emptyset$, then $\beta^* = \beta$.*

Definition 2.6. Let $\phi : H \longrightarrow H/\beta^*$ be the fundamental map of a Hv-group then, the kernel of ϕ is called ***core*** and is denoted by ω_H. Moreover, H is called $|\omega_H|$-Hv-group.

Definition 2.7. Let $(H, \cdot)$ be a Hv-structure.[5] An element $e \in H$ is called ***identity*** if $x \in ex \cap xe, \forall \in H$. We write E the set of all identities of $(H, \cdot)$. We define analogously the ***left (right) identity***.

Definition 2.8. An element x of a Hv-structure is called ***complete*** if $xy = yx = \beta^*(xy), \forall y \in H$. We define analogously the ***left (right) complete element***. Thus, if all the elements of a Hv-group are complete, then the Hv-group is *complete in the sense of Corsini*[5].

Definition 2.9. The Hv-semigroup $(H, \cdot)$ will be called h/v-group if H/β^* is a group[17].

Definition 2.10. ***Simple product*** of $x_1, x_2, ..., x_n$, is called every hyperproduct of these elements in a given order. It is denoted $p(x_1, x_2, ..., x_n)$ and the set of all corresponding simple products by $P(x_1, x_2, ..., x_n)$. If $x_1 = x_2 = ... = x_n$ we abbreviate by $p_n(x)$ and $P_n(x)$ corespondingly. The **powers** of an element x are the following: $x^1 = x, x^n = \bigcup_{p_n \in P_n} p_n(x)$, n>1. One can find the necessary definitions and properties about cyclicity in [1] and[16].

3. s_1-hyperstructures and s_1-Hv-structures

One can find the following definitions and propositions in [1], [2],and [3].

Definition 3.1. s_1-hyperstructure is the hyperstructure that all its fundamental classes are singleton but one.

These hyperstructures are denoted by the prefix "s_1", for example s_1-Hv-group, s_1-subhypergroup, where "s" means single and "1" denotes that only one fundamental class has at least two elements. The s_1-hyperstructures are, in a manner, relevant to the Very Thin Hyperstructures [15],[16]regarding their fundamental classes.

Definition 3.2. H is a s_1-Hv-group (s_1-h/v-group or s_1-hypergroup) of type (I) or type (II) iff $|\omega_H| = 1$ or $|\omega_H| > 1$ respectively.

Proposition 3.3. *If $(H, \cdot)$ is an s_1-Hv-group of type (I) then $(H, \cdot)$ is a complete 1-hypergroup.*

Proposition 3.4. *Let $(H, \cdot)$ be an s_1-Hv-group of type (I) and set S_H finite, then there exists, up to isomorphism, as many s_1-Hv-groups as the groups of order $|S_H| + 1$.*

Proposition 3.5. *Let $(H, \cdot)$ be a finite s_1-Hv-group of type (II). There exist, up to isomorphism, as many s_1-Hv-groups as the groups of order $|S_H| + 1$ multiplied by the number of different, up to isomorphism, Hv-groups of order $|H| - |S_H|$.*

4. s_2-hyperstructures and s_2-Hv(h/v)-structures

We recall[16] that if $(H, \cdot)$ is a Hv-group then every hyperproduct which has at least one single element as a factor is equal to the corresponding fundamental class. In other words this element is a complete one.

Proposition 4.1. *Let $(H, \cdot)$ be a Hv-semigroup and exist (a,b,c)$\in H^3$ such that (ab)c or a(bc) is single. Then, (ab)c=a(bc).*

Proof. Let $(H/\beta^*, \circ)$ be theβ^* fundamental semigroup of H and (ab)c=s$\in S_H$. Then, $(\beta^*(a) \circ \beta^*(b)) \circ \beta^*(c) = \{s\}$. Hence, $\beta^*(a) \circ (\beta^*(b) \circ \beta^*(c)) = \{s\}$. It is also valid that $a(bc) \subseteq \beta^*(a) \circ ((\beta^*(b) \circ \beta^*(c)) \Rightarrow a(bc) = \{s\} \Rightarrow a(bc) = (ab)c$. □

Proposition 4.2. *Let $(H, \cdot)$ be a Hv-group and one, at least, of a, b, c is a complete element. Then, (ab)c = a(bc).*

Proof. Let a be a complete element then $(ab)c = \beta^*(ab) \cdot c = \beta^*(z), z \in (ab)c$. Similarly, $a(bc) = \beta^*(z'), z' \in a(bc)$. Since $(ab)c \cap a(bc) \neq \emptyset$, it is valid $\beta^*(z) = \beta^*(z') \Rightarrow (ab)c = a(bc)$. The proof is similar if b or c is single. □

Remark 4.3. Moreover, proposition 4.2 is valid when a or c are, respectively, the left or the right complete element.

Definition 4.4. s_2-hyperstructure and s_2-Hv(h/v)-structure is every hyperstructure (Hv (h/v)-structure) that all its β^* fundamental classes are singleton except for two.

The following propositions are, obviously, valid.

Proposition 4.5. *If $(H, \cdot)$ is an s_2-Hv-group and S_H finite then $|H/\beta^*| = |S_H| + 2$. Moreover, if H is finite ($|H| = n$) then $0 < |S_H| < |H| - 3, n > 4$.*

Corollary 4.6. *If $(H, \cdot)$ is an s_2-Hv-group and $|S_H| = p - 2, p > 2$, where p is a prime then $H/\beta^* \cong Z_p$.*

Corollary 4.7. *If $(H, \cdot)$ is an s_2-Hv-group and $|S_H| = 3$ or 5 then $H/\beta^* \cong Z_5$ or Z_7 respectively.*

Corollary 4.8. *If $(H, \cdot)$ is an s_2-Hv-group and $|S_H| = 4$ then $H/\beta^* \cong Z_6$ when $(H, \cdot)$ is COW, otherwise $H/\beta^* \cong S_3$.*

Proof. $|S_H| = 4 \Rightarrow |H/\beta^*| = 6$. If $(H, \cdot)$ is COW then its fundamental group is abelian of order 6, so $H/\beta^* \cong Z_6$. If is non commutative, then $H/\beta^* \cong S_3$. □

Let $(H, \cdot)$ be an s_2-hypergroup (Hv (h/v)-group). Its β^* fundamental group is the union of two classes H_1, H_2 with, at least, two elements each and the set S_H with, at least, one single element. Hence, $H = S_H \cup H_1 \cup H_2$, $S_H = \{s_1, s_2, ...\}$, $H_1 = \{x_1, x_2, ...\}$, $H_2 = \{y_1, y_2, ...\}$ where $|S_H| > 0$, $|H_1| > 1$ and $|H_2| > 1$.

The core ω_H is the unitary element of H/β^*. Then, we can sort these structures in the order of their core. In particular, we study the following cases: The core is a singleton (case I) or not (case II).

Let $(H, \cdot)$ be an s_2-hypergroup (Hv (h/v)-group).

Definition 4.9. $(H, \cdot)$ is an s_2-hyperstructure (strong or weak) of type (I) if its core is a singleton, otherwise $(H, \cdot)$ is an s_2-hyperstructure (strong or weak) of type (II).

Proposition 4.10. *If $(H, \cdot)$ is an s_2-Hv-group of type (I) then $H_1 \cdot H_2$ and $H_2 \cdot H_1$ are single elements.*

Proof. $(H, \cdot)$ is an s_2-hyperstructure of type (I), consequently there exists a single element e such that $\omega_H = \{e\}$. The factors of the hyperproduct $H_1 \cdot H_2$ are fundamental classes, so the hyperproduct, thanks to reproductivity of classes (RC) [16], is a fundamental class too. Let us suppose that $|H_1 \cdot H_2| > 1$, for example $H_1 \cdot H_2 = H_1$. Then $\omega_H = H_2 \Rightarrow |H_2| = 1$, which is absurd. Thus, $H_1 \cdot H_2 = s \in S_H$. The proof for $H_2 \cdot H_1$ is similar. □

Under the above proposition we can distinguish the following three different types.

Definition 4.11. (Classification) An s_2-hyperstructure (strong or weak) of type (I) is:

s_2-hyperstructure of type (I_a) if $(H_1)^2$ and $(H_2)^2$ are single elements,
s_2-hyperstructure of type (I_b) if only one of $(H_1)^2$ and $(H_2)^2$ is single element,
s_2-hyperstructure of type (I_c) if $(H_1)^2 = H_2$ and $(H_2)^2 = H_1$.

Proposition 4.12. *If $(H, \cdot)$ is an s_2-Hv-group of type (II) with $\omega_H = H_1$, then $(H_2)^2 = H_1$ or s, $s \in S_H$.*

Proof. Let $\omega_H = H_1$. Then (RC) $H_1 \cdot H_2 = H_2 \cdot H_1 = H_2$. Therefore, thanks to reproductivity of classes, $(H2)^2 = H_1$ is valid, otherwise $(H2)^2 = s \in S_H$. □

In analogous manner to the above, we can distinguish the following two different types.

Definition 4.13. (Classification) An s_2-hyperstructure (strong or weak) of type (II) is:

s_2-hyperstructure of type (II_a) if $(H_2)^2 = H_1$,
s_2-hyperstructure of type (II_b) if $(H_2)^2$ is a single element.

Proposition 4.14. *If $(H, \cdot)$ is an s_2-Hv-group of type (Ia) then $(H, \cdot)$ is a complete 1-hypergroup.*

Proof. All single elements are complete. Since H is of type (Ia) it is valid

$$xy = u \in S_H, \forall (x, y) \in H_i \times H_j, i, j \in \{1, 2\}.$$

Therefore $xy = \beta^*(xy), \forall (x, y) \in H_2$. We also have that $\omega_H = \{e\}$, so $(H, \cdot)$ is a complete 1-hypergroup in the sense of Corsini[5]. □

Let $(H, \cdot)$ be an s_2-Hv-group of type (Ia) with finite set of single elements. The following proposition shows us the way to enumerate s_2-Hv-groups of type (Ia) if S_H is finite.

Proposition 4.15. *Let $(H, \cdot)$ be an s_2-Hv-group of type (Ia) and S_H finite. There exist, up to isomorphism, as many s_2-Hv-groups as the groups of order $|S_H| + 2$.*

Proof. [3],[4] Every hyperproduct of two elements is equal to the corresponding fundamental class. Let us consider the table of the hyperoperation and remove all the elements of H_1 and H_2 except for one of each class. Thus, we obtain the table of H/β^* which is the operation-table of a group of order $|S_H| + 2$. □

Proposition 4.16. *If $(H, \cdot)$ is an s_2-Hv-group of type (Ib) then $(H, \cdot)$ is an 1-hypergroup.*

Proof. We have to prove that $(\cdot)$ is (strong) associative. Since all the elements of $S_H \cup H_2$ are complete, one needs to check only the triples of elements of H_1 [4.2.]. Let $x, y, z \in H_1$, then $x(yz) = x \cdot Y_{ij} \subseteq x \cdot H_2 = s_i = \beta^*(a), a \in x(yz)$. On the other hand, $(xy)z = Y_{ij} \cdot z \subseteq H_2 \cdot z = s_j = \beta^*(b), b \in (xy)z$. As $x(yz) \cap (xy)z \neq \emptyset$ we obtain $\beta^*(a) = \beta^*(b) \Rightarrow x(yz) = (xy)z, \forall x, y, z \in H_1$. It is also valid that $\omega_H = \{e\}$ so $(H, \cdot)$ is a 1-hypergroup. $\square$

A direct conclusion of the above proposition is the proposition below which shows how "quickly" the β^* fundamental class is completed. One denotes that s_2-Hv-group of type (Ib) is *complete like.*

Proposition 4.17. *If $(H, \cdot)$ is an s_2-Hv-group of type (Ib) then, every simple product of three elements of H is the corresponding fundamental class.*

$$p(x, y, z) = \beta^*(a), a \in p(x, y, z), \forall x, y, z \in H.$$

Let $(H, \cdot)$ be an s_2-Hv-group of type (Ic), $x_i \cdot x_j = Y_{ij} \subseteq H_2, x_i, x_j \in H_1$ and $y_i \cdot y_j = X_{ij} \subseteq H_1, y_i, y_j \in H_2$. Thanks to the reproduction axiom we have that $\bigcup_{i,j\in N}(Y_{ij}) = H_2$, $\bigcup_{i,j\in N}(X_{ij}) = H_1$, $\forall i, j \in N$.

As in the s_2-Hv-groups of type (Ib) so in type (Ic) the reproductivity of every element generates an Hv-group and the reproductivity of β^* classes generates an h/v-group. Since all the finite hyperproducts, except X_{ij}, Y_{ij}, are the corresponding β^* classes, we restrict our study to $H_1 \cdot H_2, H_2 \cdot H_1$.

Proposition 4.18. *If $(H, \cdot)$ is an s_2-Hv (h/v)-group of type (Ic) and $\omega_H = \{e\}$, then $xy = yx = e, \forall x, y \in H_1 \times H_2, H_2 \times H_1$.*

Proof. Let $a, x \in H_1$ and $y \in H_2$, then $a(xy) = a \cdot s_i, s_i \in S_H$. We have that $ae = H_1$ and $H_2 = a \cdot H_1$, so, because of the reproduction axiom, $a \cdot s_i = s \in S_H, \forall s_i \in S_H - \{e\}$ is valid. Therefore, if $xy \neq e$ then $a(xy) = s \in S_H$. It is also valid that $(ax)y = Y_{ij} \cdot c \subseteq H_2 \cdot c \subseteq H_1$. Nevertheless $(\cdot)$ is WASS and $a(xy) \cap (ax)y \neq \emptyset \Rightarrow S_H \cap H_1 \neq \emptyset$, which is absurd. Thus, $xy = e$. In a similar way $(yx)a \cap y(xa) \neq \emptyset \Rightarrow yx = e$. $\square$

Let $(H, \cdot)$ be an s_2-Hv-group of type (IIa) i.e.
$\omega_H = H - 1$, $x_i \cdot x_j = X_{ij} \subseteq H_1, x_i, x_j \in H_1$,$y_i \cdot y_j = X'_{ij} \subseteq H_1, y_i, y_j \in H_2$, $x_i \cdot y_j = Y_{ij} \subseteq H_2, x_i \in H_1, y_j \in H_2$ and $y_i \cdot x_j = Y'_{ij} \subseteq H_2, x_j \in H_1, y_i \in H_2$. Thanks to the reproduction axiom we have that

$$\bigcup_{i,j\in N}(Y_{ij}) = \bigcup_{i,j\in N}(Y'_{ij}) = H_2 \text{ and}$$
$$\bigcup_{i,j\in N}(X_{ij}) = \bigcup_{i,j\in N}(X'_{ij}) = H_1, \ \forall i, j \in N.$$

Proposition 4.19. *If $(H, \cdot)$ is an s_2-Hv-group of type (IIa), then $(H_1 \cup H_2, \cdot)$ is a sub-Hv-group of H such that $(H_1 \cup H_2)/\beta^* \cong Z_2$.*

Proof. It is obvious that $xy \subseteq H_1 \cup H_2, \forall x, y \in H_1 \cup H_2$. Since $H_1 \cap H_2 = \emptyset$ and the reproduction axiom is valid, we obtain that $x(H_1 \cup H_2) = xH_1 \cup xH_2 = H_1 \cup H_2$ or $H_2 \cup H_1$ if $x \in H_1$ or $x \in H_2$ respectively. We, similarly, prove that $(H_1 \cup H_2)x = H_1 \cup H_2, \forall x \in H_1 \cup H_2$. Therefore, $(H_1 \cup H_2, \cdot)$ is a sub-Hv-group of H.
It is also valid that $(H_1 \cup H_2)/\beta^* = \{H_1, H_2\}$, so the fundamental group is isomorphic to Z_2. $\square$

Definition 4.20. Set K is called ***sub-h/v-group*** of H if H is an h/v-group, $\emptyset \neq K \subseteq H, K.K \subseteq K$ and K/β^* is a group.

Remark 4.21. It is obvious that analogous proposition to the above one is valid if $(H,\cdot)$ is an s_2-h/v-group of type (IIa), where $(H_1 \cup H_2, \cdot)$ is a sub-h/v-group of H. This is so, because only among the elements of $H_1 \cup H_2$ the reproduction axiom may not be valid.

Let $(H,\cdot)$ be an s_2-Hv-group of type (IIb) i.e.
$\omega_H = H_1$, $x_i \cdot x_j = X_{ij} \subseteq H_1, x_i, x_j \in H_1$ $,y_i \cdot y_j = u \in S_H, y_i, y_j \in H_2$,
$x_i \cdot y_j = Y_{ij} \subseteq H_2, x_i \in H_1, y_j \in H_2$ *and* $y_i \cdot x_j = Y'_{ij} \subseteq H_2, x_j \in H_1, y_i \in H_2$.
Thanks to the reproduction axiom we have that

$$\bigcup_{i,j\in N}(Y_{ij}) = \bigcup_{i,j\in N}(Y'_{ij}) = H_2 \text{ and}$$
$$\bigcup_{i,j\in N}(X_{ij}) = H_1,\ \forall i,j \in N.$$

Remark 4.22. In the special case where there is an extra condition such that $u \cdot H_2 = H_2 \cdot u = H_1$ and $u^2 = H_2$, then $(K,\cdot)$ is an s_2-sub-Hv-group of H, where $K = H_1 \cup H_2 \cup \{u\}$. We also have that $K/\beta^* \cong Z_3$.

The proof is the same as the one of proposition 4.19. Analogous proposition to the above one is valid if $(H,\cdot)$ is an s_2-h/v-group of type (IIb).

Proposition 4.23. *Let $(H,\cdot)$ be an s_2-Hv-group of type (II) and $|S_H| = 2$, then $H/\beta^* \cong (Z_2)^2$ if it is of type (IIa) and $H/\beta^* \cong Z_4$ if it is of type (IIb).*

Proof. Since [4.5] H/β^* is a group of order 4 we deduce that the fundamental group is the cyclic group of order 4 (C_4) or the Klein 4-group.
Let $(H,\cdot)$ be an s_2-Hv-group of type (IIa), $S_H = \{s,u\}$ and V the Klein four group presented as $< a,b; a^2 = b^2 = (ab)^2 = 1 >$.
Let $f : H/\beta^* \to V$ be the following map: $f(H_1) = 1, f(H_2) = a, f(s) = b, f(u) = ab$. It is trivial to prove that f is an isomorphism such that $H/\beta^* \cong (Z_2)^2$.
In a similar way, if $(H,\cdot)$ is an s_2-Hv-group of type (IIb) and $S_H = \{s,u\}$, one takes the isomorphism $g : g(H_1) = 1, g(H_2) = a, g(s) = a^2, g(u) = a^3$, where $C_4 :< a; a^4 = 1 >$. Then $H/\beta^* \cong Z_4$. □

The following proposition presents a construction of an s_2-Hv-group in order to have a given fundamental group. This construction is based on Construction S_3[16],[17].

Proposition 4.24. ***(Construction** $s_2 - H$**).*** *Let $(G,\cdot)$ be a group. Consider any pairwise disjoint family of sets $\{H_g / g \in G\}$ with indexes from G.*
Assume that

(i) $g \in H_g, \forall g \in G$,
(ii) $\exists x,y \in G : x \neq y$ *and* $|H_x|, |H_y| > 1$
(iii) $|H_g| = 1, \forall g \in G - \{x,y\}$.

Then the set $H = \bigcup_{g\in G}(H_g)$ with respect to every hyperoperation $(\circ)$ such that

$$gh \subseteq a \circ b \subseteq H_{gh}, \forall (a,b) \in H_g \times H_h, \forall g,h \in G \text{ and}$$
$$a \circ H_h = H_{gh}, H_h \circ a = H_{hg}, \forall a \in H_g, \forall g,h \in G,$$

is an s_2-Hv-group with $H/\beta^* \cong G$.

Proof. The proof of the proposition is the same as in [17](Theorem 11). □

The type of the s_2-Hv-group which is constructed using the above proposition 4.24, depends on the elements x, y and the relation between the powers x^2 and y^2 where $|H_x|, |H_y| > 1$. Let $(H, \circ)$ be an s_2-Hv-group resulting from the construction 4.24 and e the unitary element of $(G, \cdot)$. Considering the definitions 4.11 and 4.13 we obtain, in an obvious way, the following propositions.

Proposition 4.25. *$(H, \circ)$ is an s_2-Hv-group of type (I) if $x, y \neq e$. Otherwise, if $e \in \{x, y\}$ then $(H, \circ)$ is an s_2-Hv-group of type (II).*

Corollary 4.26. ***(Criteria for type (I))***
Let $(H, \circ)$ be an s_2-Hv-group and $e \notin \{x, y\}$. Then $(H, \circ)$ is:

S_2-Hv-group of type (Ia) if $x^2 \neq y$ and $y^2 \neq x$,
S_2-Hv-group of type (Ib) if $x^2 = y$ and $y^2 \neq x$,
S_2-Hv-group of type (Ic) if $x^2 = y$ and $y^2 = x$.

Remark 4.27. $(x^2 = y$ and $y^2 = x.) \Leftrightarrow xy = e$.

Proof. $x^2 = y \Leftrightarrow x^4 = y^2 \Leftrightarrow x^4 = x \Leftrightarrow x^3 = e \Leftrightarrow xx^2 = e \Leftrightarrow xy = e$. □

The above remark explains why proposition 4.18 holds.

Corollary 4.28. ***(Criteria for type (II))***
Let $(H, \circ)$ be an s_2-Hv-group and $x = e$. Then $(H, \circ)$ is:

S_2-Hv-group of type (IIa) if $y^2 = e$,
S_2-Hv-group of type (IIb) if $y^2 \neq e$.

Example 4.29. Let $G = \{e, s, t, x, y\}$ and $(G, \cdot) \cong Z_5 =< s >$ where e is the unitary element, $x = s^3, y = s^4$. Let $H = \{e, s, t, x, a, b, y, c, d\}$ where

$$H - e = \{e\}, H_s = \{s\}, H - t = \{t\}, H_x = \{x, a, b\} \text{ and } H_y = \{y, c, d\}.$$

The hyperstructure $(H, \circ)$ of which the Cayley table is the table 1 is an s_2-Hv-group of type (Ib) because $x^2 = s^6 = s \neq y$ and $y^2 = s^8 = s^3 = x$, [4.24].

Table 1

∘	**e**	**s**	**t**	**x**	**a**	**b**	**y**	**c**	**d**
e	e	s	t	x,a,b	x,a,b	x,a,b	y,c,d	y,c,d	y,c,d
s	s	t	x,a,b	y,c,d	y,c,d	y,c,d	e	e	e
t	t	x,a,b	y,c,d	e	e	e	s	s	s
x	x,a,b	y,c,d	e	s	s	s	t	t	t
a	x,a,b	y,c,d	e	s	s	s	t	t	t
b	x,a,b	y,c,d	e	s	s	s	t	t	t
y	y,c,d	e	s	t	t	t	x,a,b	x	x
c	y,c,d	e	s	t	t	t	x	x,a,b	x
d	y,c,d	e	s	t	t	t	x	x	x,a,b

Example 4.30. (s_2-h/v-group)
Let $H = \{a, b, c, d, f, g\}$ be equipped by the hyperoperation $(\cdot)$ of which the Cayley table is **table 2**. Let $H_1 = \{a, b\}, H_2 = \{b, c\}$. It is trivial to find that f and g are single elements, so $S_H = \{f, g\}$.

Table 2

$\cdot$	**a**	**b**	**c**	**d**	**f**	**g**
a	a, b	a	c, d	c	f	g
b	a	b	c	c	f	g
c	c	c, d	a, b	a, b	g	f
d	c, d	c	a, b	a, b	g	f
f	f	f	g	g	a, b	c, d
g	g	g	f	f	c, d	a, b

Proposition 4.31. *$(H.\cdot)$ is an s_2-h/v-group of type (IIa) such that $H/\beta^* \cong (Z_2)^2$.*

Proof. **Table 2** shows that the reproduction axiom is not valid for b and d because,

$$b \cdot H = H - \{d\} \subset H = H \cdot b \quad and \quad H \cdot d = H - \{d\} \subset d \cdot H = H.$$

As $(H, \cdot)$ is COW, the cases one has to check the WASS are [16](p.30), the triples $(x, x, y), (x, y, y), (y, x, x), (y, y, x)$ for $x \neq y$ and every triple with three different pairwise elements of H. Checking the triples we have:

(i) the triples where $(xy)z \subset x(yz)$ are $(aa)d, (ab)d, (ad)a, (ad)c, (ad)d, (bc)c, (bc)d, (bd)c, (bd)d, (ca)a, (ca)c, (ca)d, (cc)d, (cd)d, (db)a, (dc)d, (dd)d, (ff)d, (gg)d$.
(ii) the triples where $(xy)z \supset x(yz)$ are $(ba)c, (bc)b, (bd)b, (ca)b, (cb)a, (db)b$.
(iii) all the rest checked triples are such that $(xy)z = x(yz)$.

Thus, $(H, \cdot)$ is an s_2-Hv-semigroup of type (IIa) because it is WASS and $\omega_H = H_1$.

Table 3

$\bullet$	H_1	H_2	$\{f\}$	$\{g\}$
H_1	H_1	H_2	$\{f\}$	$\{g\}$
H_2	H_2	H_1	$\{g\}$	$\{f\}$
$\{f\}$	$\{f\}$	$\{g\}$	H_1	H_2
$\{g\}$	$\{g\}$	$\{f\}$	H_2	H_1

The fundamental classes of H are $H_1 = \{a, b\}, H_2 = \{c, d\}, \{f\}$ and $\{g\}$.
Table 3 is the Cayley table of H/β^*. It is obvious that the reproductivity of classes is valid, so H/β^* becomes a group.
Since $(H, \cdot)$ is an Hv-semigroup we conclude that $(H, \cdot)$ is a h/v-group.
A presentation for the Klein 4 group is $< x, y; x^2 = y^2 = (xy)^2 = 1 >$.
Let us consider the map
$h : H/\beta^* \rightarrow \{1, x, y, xy\} : h(H_1) = 1, h(H_2) = x, h(\{f\}) = y$ and $h(\{g\}) = xy$.
Obviously, h is an *onto* and $1:1$ homomorphism, so $H/\beta^* \cong (Z_2)^2$. □

The powers of the elements of H and its sub-hyperstructures are studied in the following propositions.

Proposition 4.32. *The powers of the elements of H are the following:*

$$a^n = \omega_H, \forall n > 1, \quad b^n = b, \forall n > 0, \quad c^{2n} = d^{2n} = f^{2n} = g^{2n} = H_1,$$
$$c^{2n+1} = d^{2n+1} = H_2 \text{ and } f^{2n+1} = f, g^{2n+1} = g, \forall n > 0.$$

Proof. Notice that $a^2 = \omega_H, b^2 = b, t^2 = H_1$ where $t \in \{c, d, f, g\}, c^3 = d^3 = H_2$ and $f^3 = f, g^3 = g$. The proof is carried out using mathematical induction. □

Figure 1 shows the lattice of sub-h/v-groups of H (in the notation of the Cayley table above).

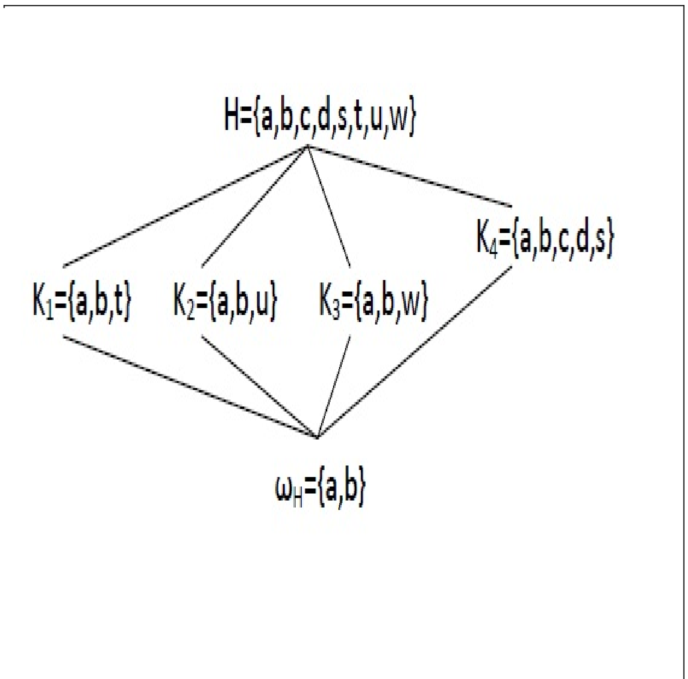

Figure 1

Remark 4.33. Notice that ω_H is hypergroup, K_1 is sub-h/v-group and K_2, K_3 are s_1-hypergroups of type (II). It is also valid that $K_i \cong Z_2, i = 1, 2, 3$. Regarding cyclicity we mention that:

(i) K_1 is cyclic h/v-group of period 3 and set of generators the class H_2,
(ii) K_2 and K_3 are cyclic hypergroups of period 2 and generators f and g respectively,
(iii) ω_H is a single-power cyclic hypergroup and element a is the generator of single-power period 2.

Example 4.34. Let **table 4** be the Cayley table of group $(G, \cdot), G = \{1, 2, s, t, u, v\}$ which is isomorphic to S_3, the symmetric group of degree 3.

Table 4

·	**1**	2	**s**	**t**	**u**	**w**
1	1	2	s	t	u	w
2	2	s	1	u	w	t
s	s	1	2	w	t	u
t	t	w	u	1	s	2
u	u	t	w	2	1	s
w	w	u	t	s	2	1

Consider the sets $H_1 = \{a,b\}, H_2 = \{c,d\}, H_s = \{s\}, H_t = \{t\} H_u = \{u\}$, $H_v = \{v\}$ *and* $H = \{a,b,c,d,s,t,u,v\}$, *where* $a = 1, c = 2$.
Set H *equipped by a hyperoperation under construction 4.24 is a Hv-group. The Cayley table of the resulting Hv-group is* ***table 5***.

Table 5

$\circ$	**a**	**b**	**c**	**d**	**s**	**t**	**u**	**v**
a	a	a,b	c	c,d	s	t	u	v
b	a,b	a	c,d	c	s	t	u	v
c	c	c,d	s	s	a,b	u	v	t
d	c,d	c	s	s	a,b	u	v	t
s	s	s	a,b	a,b	c,d	v	t	u
t	t	t	v	v	u	a,b	s	c,d
u	u	u	t	t	v	c,d	a,b	s
v	v	v	u	u	t	s	c,d	a,b

Proposition 4.35. $(H,\circ)$ *is an* s_2*-Hv-group of type* (IIb) *such that* $H/\beta^* \cong S_3$.

Proof. Construction 4.24 ensures that $(H,\circ)$ is Hv-group such that $H/\beta^* \cong S_3$. Since, $S_H = \{s,t,u,v\}$, and $|H_1| = |H_2| = 2$, then $(H,\circ)$ is an s_2-Hv-group. Moreover, $a = 1$ and $c^2 = s \neq 1$, so [4.28] $(H,\circ)$ is an s_2-Hv-group of type (IIb). □

Figure 2 shows the lattice of sub-Hv-groups of H (in the notation of the Cayley table).

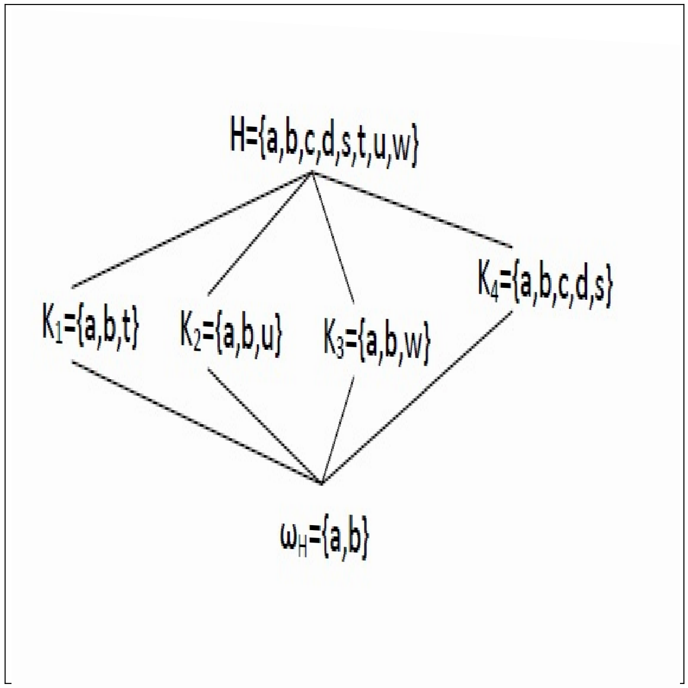

Figure 2

Remark 4.36. N[illegible] of type (II) and K_4 is s_2-Hv-group of type (IIb). It is also valid that $K_i \cong Z_2, i = 1, 2, 3$ and $K_4 \cong Z_3$. All these sub-Hv-groups are commutative.

Proposition 4.37. *The powers of the elements of H have the following properties:*

$$a^n = a, \forall n > 0, \quad b^n = \omega_H, \forall n > 2, \quad c^n = d^n = K_4, \forall n > 7,$$
$$s^{3n} = H_1, \quad s^{3n+1} = H_2, \quad s^{3n+2} = s, \forall n > 0, \text{ and}$$
$$x^{2n} = H_1, \quad x^{2n+1} = x, \forall n > 0, \quad \forall x \in \{t, u, v\}.$$

Proof. Studying all the cases we find some small powers of all elements and then we are completing the proof by the mathematical induction. Thus we have:
$a^2 = a, b^2 = a$ and $b^3 = \omega_H, c^2 = s, c^3 = \omega_H, c^4 = \{a, b, c, d\}, c^5 = \{c, d, s\}$,
$c^6 = \{a, b, s\}, c^7 = c^4 = \{a, b, c, d\}$ and $c^8 = \{a, b, c, d, s\}$ which is a sub-Hv-group of H. The powers of d are the same with c, as $d^n = c^n, \forall n > 1$.
The powers of s are $s^2 = \{c, d\}, s^3 = \{a, b\}, s^4 = s$ and $s \cup s^2 \cup s^3 = \{a, b, c, d, s\}$ which is a sub-Hv-group of H.
We also have that $x^2 = \{a, b\} = \omega_H, x \cup x^2 = \{a, b, x\}$ which is a sub-Hv-group of $H, \forall x \in \{t, u, v\}$. □

Remark 4.38. Regarding cyclicity of H we mention that:

(i) ω_H is a single-power cyclic Hv-group and b is the generator of single-power period 2,

(ii) K_1, K_2 and K_3 are cyclic Hv-groups of period 2 and generators t, u and v respectively and

(iii) K_4 is cyclic Hv-group with generator s of period 3 and generators c, d of period 4. K_4 is also single-power cyclic where c, d are the generators of single-power period 8.

We give the following definition for the general case, which is an enlargement of previous definitions of s_1 and s_2-hyperstructures.

Definition 4.39. We shall say that a hyperstructure (Hv (h/v)-structure) is an s_n***-hyperstructure (***s_n***-Hv (h/v)-structure)*** if all its β^* fundamental classes are singleton except for n of them, $n \in \mathbb{N}$.

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